ARMAX MODAL PARAMETER ESTIMATION USING RANDOM AND PERIODIC EXCITATION

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Abstract

Experimental methods used to determine the dynamic behaviour of helicopter structures are broadly categorised as either ground testing methods or in-flight testing methods. A major limitation of ground testing is that the boundary conditions of the grounded helicopter structure affect the results. Therefore, it is desirable to carry out testing while the helicopter is in flight. Recently, a number of time-domain parameter estimation methods have been proposed that estimate modal parameters from response-only measurements. A significant assumption used in these methods is that the unmeasured excitation approximates white noise. For the case of a helicopter in flight, rotor hub loads as well as pressure pulses from passing blades induce periodic excitations. The aim of this study is to investigate the capabilities of a time domain parameter estimation technique applied to structures subject to unmeasured periodic excitation and measured random excitations.

An autoregressive moving average with exogenous excitation (ARMAX) model is considered, which requires that a component of the excitation be measured. Parameter estimation of the ARMAX model is carried out using an iterative multistage method and a two degree-of-freedom numerical simulation is used to generate excitation and response data corrupted with noise. The effects of different relative levels of random and unmeasured periodic excitation are studied. Results show that the algorithm accurately estimates modal frequencies for all cases of noise. Unmeasured periodic excitations were shown to affect the accuracy of damping estimates and mode shapes.

1 Introduction

The design, modification, and maintenance of helicopters is dependent on accurate knowledge of the helicopter structure’s dynamic properties. This information can be used to validate and update finite element models [1, 2], study fatigue of structural components, and predict component life times. A number of studies have shown that classical input-output modal analysis techniques can yield useful information and have characterised the complexity and non-linear nature of helicopter structures [3-7]. Further investigation of the dynamic properties of helicopter structures during flight has shown significant variations in the measured properties compared to ground testing results [8-10]. This suggests the need for operational modal analysis techniques that can be applied to helicopters in flight.

A major difficulty with applying existing operational modal analysis techniques to helicopters in flight is significant periodic excitations induced by the main and tail rotors [5, 11]. In addition to these loads, the pressure pulses caused by the down-wash from the main rotor act locally on the
It is envisaged that a combination of input-output modal analysis and operational modal analysis techniques can be used to overcome these difficulties and yield a full set of modal data for a helicopter structure in flight. This includes an attempt to obtain scaled mode shapes, which are not typically calculated by operational modal analysis methods.

The proposed modal analysis methods are based on system identification techniques that estimate parameters of a mathematical model directly from time-domain data. An autoregressive moving average with exogenous excitation (ARMAX) model is used in the case of input-output modal analysis and a corresponding autoregressive moving average (ARMA) model is used for the case of output only modal analysis. Estimation of the parameters of these models is based on a multistage method proposed by Fassois et al. Estimation of the ARMA model yields the global modal parameters and unscaled mode shapes and the ARMAX model can be used to estimate the global modal parameters and scaled mode shapes.

The following section outlines the development of the ARMAX model structure and the estimation procedure. Numerical testing results are discussed in section three and concluding remarks are included in section four.

## 2 ARMAX Model Structure and Estimation of Modal Parameters

The ARMAX model structure can be defined as follows [17]

\[
A(q) \cdot y[t] = B(q) \cdot f[t] + C(q) \cdot w[t], \tag{1}
\]

where

\[
A(q) = I_s + A(1) \cdot q + \cdots + A(\text{na}) \cdot q^{\text{na}}, \quad [s \times s] \tag{2}
\]

\[
B(q) = B(0) + B(1) \cdot q + \cdots + B(\text{nb}) \cdot q^{\text{nb}}, \quad [s \times m] \tag{3}
\]

\[
C(q) = C(1) \cdot q + \cdots + C(\text{nc}) \cdot q^{\text{nc}}, \quad [s \times s] \tag{4}
\]

\[f[t] \ [m \times 1] \] is the excitation vector and \( y[t] \ [s \times 1] \) is the response vector, which is corrupted by random measurement noise. \( A(q), B(q), C(q) \) are the autoregressive (AR), exogenous (X), and moving average (MA) matrix polynomials, respectively. The dimensions of the matrix coefficients are determined by the number of response measurements, \( s \), and the number of excitation measurements, \( m \). \( I_s \) is the \([s \times s]\) identity matrix and \( q \) is the backshift operator such that \( x[t] \cdot q = x[t-1] \); \( x[t] \cdot q^j = x[t-j] \). \( w[t] \) is an unobservable zero-mean white noise sequence. The estimation of the coefficients of the AR, X, and MA matrices is based on work by Fassois et al [15, 16], however, a number of modifications to the procedure have been adopted. The algorithm is a multistage method based around the least-squares solution of ARX models and subsequent filtering of the input and output data. Fassois et al discussed many advantages of this approach and also demonstrated its effectiveness in a number of numerical and experimental studies [15, 16].

### 2.1 Stage 1

Stage 1 involves estimation of a higher order ARX model. The following ARX model can be derived from equation (1) [15].

\[
H_r(q) \cdot y[t] = H_r(q) \cdot f[t] + w[t], \tag{5}
\]

where

\[
H_r(q) = \frac{B(q)}{A(q)} \cdot \frac{C(q)}{A(q)} \cdot \frac{1}{1 - R(q)},
\]
\[
H_f(q) = \mathbf{I} + \sum_{j=1}^{s} \mathbf{H}_f(j) \cdot q^j = \mathbf{C}^{-1}(q) \cdot \mathbf{A}(q), \quad [s \times s] \tag{6}
\]
\[
H_y(q) = \sum_{j=1}^{s} \mathbf{H}_y(j) \cdot q^j = \mathbf{C}^{-1}(q) \cdot \mathbf{B}(q), \quad [s \times m]. \tag{7}
\]

Ljung [17] noted that as the number of data \(N\) and the model order \(p\) approaches infinity \((N > p)\) the transfer function \(H(q)/H_y(q)\) will converge to that of the ‘true system’. In this case, the true system includes noise characteristics and so a higher-order model is selected to account for this. Equation (5) is rewritten in terms of the regression vector \(\mathbf{u}_k[t], t = 1, \ldots, N,\)

\[
\mathbf{u}_k[t] = [\mathbf{y}_k[t-1] - \mathbf{y}_k[t-2] \cdots - \mathbf{y}_k[t-p] : \mathbf{f}^T[t] \cdots \mathbf{f}^T[t-p]]^T, \quad [(p + p.m + m) \times 1] \tag{8}
\]

where \(\mathbf{y}_k[t], k = 1, \ldots, s\), is the \(k\)th row of \(\mathbf{y}[t]\), and the parameter vector \(\mathbf{h}_k\)

\[
\mathbf{h}_k = [\mathbf{H}_{yT}(1) \quad \mathbf{H}_{yT}(2) \cdots \mathbf{H}_{yT}(p) : \mathbf{H}_{yT}(0) \cdots \mathbf{H}_{yT}(p)]^T, \quad [(p + p.m + m) \times 1] \tag{9}
\]

where \(\mathbf{H}_{yT}(p)\) is the \(k\)th row of \(\mathbf{H}_y(p)\), to yield \(k\) multiple-input single-output (MISO) systems

\[
\mathbf{y}_k[t] = \mathbf{u}_k^T[t] \cdot \mathbf{h}_k + \mathbf{e}_k[t], \tag{10}
\]

which are solved using the least-squares criterion and QR factorisation as shown in Ljung [17].

In practice, the order of the elements in the input and output data vectors is reversed. A consequence of defining this ‘backwards’ system is that for moderate levels of measurement noise the poles of the true system are unstable, and therefore will be outside the unit circle on the z-plane. The remainder of the estimated poles that describe the noise components will be inside the unit circle [18]. This property provides an additional means of distinguishing between the poles of the true system and spurious numerical poles.

Separating the original multiple-input multiple-output (MIMO) system into \(k\) MISO systems leads to estimates of \(k\) scalar AR polynomials describing the global properties of the system. These are used to define an AR matrix polynomial with diagonal matrix coefficients; a coefficient of the \(k\)th scalar polynomial appears as the \(k\)th diagonal element in the corresponding matrix coefficient.

### 2.2 Stage 2

The second stage of the estimation algorithm involves calculating an estimate of \(\mathbf{C}(q)\) using the AR matrix \(\hat{\mathbf{H}}_y(q)\) \((\hat{\cdot} \text{ denotes estimate})\) estimated in the first step. Equation (6) is rewritten as [15]

\[
\sum_{j=0}^{\min(i,nx)} \mathbf{C}(j) \cdot \hat{\mathbf{H}}_y(i-j) = \mathbf{A}(i), \quad i = 0, 1, 2, \ldots \tag{11}
\]

and a system of equations can be written for \(i > \max(na, nc)\), with the approximation \(\mathbf{A}(q) = \mathbf{0}, \quad q > na\), which is solved for \(\hat{\mathbf{C}}(q)\). A further operation can be carried out to stabilise \(\hat{\mathbf{C}}(q)\), which is necessary for subsequent filtering operations. Details can be found in [15].
2.3 Stage 3

The third stage involves filtering the input and output data using \( \hat{C}(q) \) and then estimating the parameters of another ARX model. Equation (1) can be rewritten as [15]

\[
\hat{C}(q) \cdot y[t] = \hat{C}(q) \sum_{j=0}^{\infty} B(j) \cdot f[t-j] - \hat{C}(q) \sum_{j=1}^{\infty} A(j) \cdot y[t-j] + \varepsilon_2[t] \tag{12}
\]

As in the first stage, this ARX model can be separated into \( k, k=1, \ldots, s \) MISO ARX models, which in this case are in terms of the filtered response \( Y_{Fk}[t] \) and filtered excitation \( F_{Fk}[t] \):

\[
y_{Fk}[t] = \sum_{j=0}^{\infty} F_{Fk}[t-j] \cdot B_k(j) - \sum_{j=1}^{\infty} Y_{Fk}[t-j] \cdot A_k(j) + \varepsilon_{2k}[t] \tag{13}
\]

where

\[
Y_{Fk}[t] \cdot \hat{C}(q) = y_k[t] \cdot I,
\]

\[
F_{Fk}[t] \cdot \hat{C}(q) = f_k[t] \cdot I, \tag{14}
\]

\( y_{Fk}[t]= \) column \( k \) of \( Y_{Fk}[t] \), and \( y_k[t] \), \( f_k[t] \) are the \( k \)th rows of \( y[t] \) and \( f[t] \), respectively. Equations 14 and 15 show that \( \hat{C}(q) \) is implemented as an infinite impulse response (IIR) filter [19] to obtain \( Y_{Fk}[t] \) and \( F_{Fk}[t] \).

Equation (14) is rewritten in the following form and solved using the least-squares criterion and QR factorization, as outlined in Ljung [17].

\[
y_{Fk}[t] = U_{Fk}[t] \cdot \hat{\theta}_{2k} + \varepsilon_{2k}[t] \tag{16}
\]

where

\[
\hat{\theta}_{2k} = [A_k(1) \cdots A_k(na) : B_k(0) \cdots B_k(nb)]^T \quad [(na.s + nb.m.s + m.s) \times I]
\]

\[
U_{Fk}[t] = [-Y_{Fk}[t-1] \cdots -Y_{Fk}[t-na] : F_{Fk}[t] \cdots F_{Fk}[t-nb]] \tag{17}
\]

As in the first stage, the coefficients of \( k, k=1, \ldots, s \), scalar polynomials describing the global properties are obtained from \( \hat{\theta}_{2k} \) and these are combined to define the AR matrix polynomial \( A(q) \), which has diagonal matrix coefficients. Similarly, each \( B_k(q) \) is a row of the X matrix polynomial \( B(q) \).

The order of elements in the filtered excitation and response vectors is reversed as was the case in the first stage ARX model. Again, a consequence of this backwards representation is that the true system poles are unstable and will lie outside the unit circle on the z-plane. If the ARX model in equation (12) is of higher order than theoretically necessary then the extra poles will lie inside the unit circle.

2.4 Stage 4

The final stage of the estimation algorithm involves the deconvolution of \( C^{-1}(q) \) from \( \hat{H}_y(q) \) using \( \hat{A}(q) \). The definition of the convolution of two polynomials [20]

\[
H_y(k) = \sum C^{-1}(j) \cdot A(k+1-j), \quad j = \max(1, k + 1 - (na + 1)), \ldots, \min(k, p-na), \tag{19}
\]
is used to set up a system of linear equations for \( k = 1, \ldots, p \), which can be solved for \( \hat{C}^{-1}(q) \).

At this point the modal parameters may be calculated from the transfer function \( B(q)/A(q) \), however, further iterations of stages 3 and 4 can be carried out. At least one iteration is recommended as \( \hat{C}(q) \) calculated in stage two is only an approximation because of the assumption \( A(q) = 0 \). Since the final stage calculates \( \hat{C}^{-1}(q) \) and stage two calculates \( \hat{C}(q) \), the filtering operations in stage 3 described by equations 14 and 15 are rewritten as finite impulse response (FIR) filters [19]

\[
Y_{f_k}[t] = y_k[t] \cdot \hat{C}^{-1}(q) \tag{20}
\]

\[
F_{f_k}[t] = f_k[t] \cdot \hat{C}^{-1}(q) \tag{21}
\]

2.5 Stage 5

The global modal parameters are calculated by finding the eigenvalues of the companion matrix of the AR matrix polynomial. Note that since the AR and X matrices have been estimated using the backwards ARX model, the order of the matrix polynomial coefficients have to be reversed. The system natural frequencies and damping can be estimated for each pole using the following equations [21]

\[
\omega_n = \frac{1}{T_s} \sqrt{\ln z_n \cdot \ln z_n^*} \tag{22}
\]

\[
\varsigma_n = \frac{-\ln(z_n z_n^*)}{2 \cdot \omega_n \cdot T_s} \tag{23}
\]

\( T_s \) is the sampling period, \( z_n \) a complex pole of the AR matrix, and \( z_n^* \) the complex conjugate pole. The MISO model structure leads to \( s \) sets of global parameters being estimated. The system poles will typically be similar for moderate measurement noise levels and can be averaged.

The transfer function transmission zeroes are determined by calculating the roots of the polynomial \( b_y(q) \), which is formed by taking the \( ij \)th element of each matrix coefficient of \( \hat{C}(q) \).

\[
B(q) = B(0) + B(1) \cdot q + \cdots + B(nb) \cdot q^{nb}, \quad [s \times m]. \tag{24}
\]

That is,

\[
b_y(q) \equiv b_y(0) + b_y(1) \cdot q + \cdots + b_y(nb) \cdot q^{nb} \tag{25}
\]

The \( s \times m \) scalar transfer functions are factorised into partial fractions; the residues used to define the \( k \)th mode shape [15]

\[
\phi_k = \begin{bmatrix} R_{12k} & \cdots & R_{1nk} \end{bmatrix}^T \tag{26}
\]

3 Numerical Testing

Data simulating a two degree-of-freedom (DOF) discrete mass system was used to test the parameter estimation algorithm. Properties of the system are as follows:
Displacement data in the \( x \) direction was obtained at each DOF for independent zero-mean random excitations applied at each DOF. A sampling rate of 50 Hz was used and each data record was 500 samples long. Zero-mean, normally distributed measurement noise was superimposed independently on each of the response measurements; the noise level defined as the ratio of the RMS amplitude of the random noise and the RMS amplitude of the response measurement. Tests in the presence of unmeasured periodic excitations were also carried out. The periodic excitations were at 3 discrete frequencies (1 Hz, 2 Hz, 4 Hz) at a level defined by the ratio of the RMS amplitude of the summed periodic excitations and the RMS amplitude of the measured random excitations. The unmeasured periodic signal was added to the measured random excitation applied at one DOF.

The estimation algorithm was programmed to estimate a number of models with different model orders, beginning with the minimum model order required. For the MISO model structure the minimum model order required was \( na = 4 \). The order of the \( X \) matrix polynomial was set as \( nb = na - 1 \), for displacement response data [15]. A number of iterations of stages 3 and 4 were carried out and no significant changes in estimated system poles and zeros were observed after approximately 10 iterations.

Stabilisation diagrams were used to select the model of minimum order with stable system modes, although stabilisation of modal damping was difficult to identify for cases with 10% random measurement noise and cases with unmeasured periodic excitation. Development of a more reliable method for selecting model order will be considered in the future. The sign of the damping was used to indicate system poles, however, for the cases with unmeasured periodic excitations some non-system modes were estimated with positive damping. The frequencies of these modes were typically close a component of the unmeasured periodic excitation (i.e. 1, 2 or 4 Hz). Table 1 lists the estimated modal frequencies, damping, and mode shapes for each test. Mode shapes have been calculated using equation (26) so that only the magnitude and phase of the second DOF is listed.

Estimation of modal frequencies was very accurate with less than 1% deviation from true values for all tests, and this was reflected in the stabilisation diagrams with modal frequencies stabilising quickly. Modal damping was accurately estimated for the cases with random measurement noise, however, the presence of unmeasured periodic excitations affected the damping estimates of the second mode. Mode shapes, in particular the phase, were most sensitive to noise with large errors for cases with unmeasured periodic excitations.

4 Conclusion

A combination of input-output and operational modal analysis techniques has been proposed to address the problem of determining the dynamic properties of helicopter structures. These techniques are based on the identification of ARMAX/ARMA models using a multi-stage estimation algorithm adapted from [15, 16]. A MISO estimation algorithm for the ARMAX model has been outlined and data simulating the response of a two DOF system used for testing. Results show that the algorithm accurately estimates modal frequencies for all cases of noise. Unmeasured periodic excitations were shown to affect the accuracy of damping estimates and mode shapes.
<table>
<thead>
<tr>
<th>Mode</th>
<th>Model Order (na)</th>
<th>Natural Frequency (Hz)</th>
<th>Damping (%) (% Error)</th>
<th>Mode Shape (Magnitude/Phase°) (% Error)</th>
</tr>
</thead>
<tbody>
<tr>
<td>No noise</td>
<td>4</td>
<td>1.4836 (2.25E-4)</td>
<td>0.4180 (2.70E-4)</td>
<td>2.3769/0.2104/0.17899° (1.35E-4/-2.00E-6)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4.6757 (-7.57E-5)</td>
<td>1.2289 (-4.60E-6)</td>
<td></td>
</tr>
<tr>
<td>1% random measurement noise</td>
<td>10</td>
<td>1.4835 (-0.006)</td>
<td>0.4159 (-0.482)</td>
<td>2.3947/0.2105° (0.039/0.146)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4.6763 (0.013)</td>
<td>1.2427 (1.123)</td>
<td>(0.746/234.74)</td>
</tr>
<tr>
<td>10% random measurement noise</td>
<td>16</td>
<td>1.4832 (-0.027)</td>
<td>0.4121 (-1.393)</td>
<td>2.5921/0.2032° (-3.427/-1.747)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4.6722 (-0.074)</td>
<td>1.2475 (1.508)</td>
<td>(9.051/1.03e+3)</td>
</tr>
<tr>
<td>100% unmeasured periodic excitations; 1Hz, 2Hz, 4Hz</td>
<td>10</td>
<td>1.4755 (-0.548)</td>
<td>0.4060 (-2.865)</td>
<td>2.7546/0.1923° (-8.596/-4.257)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4.6780 (0.049)</td>
<td>1.1595 (-5.650)</td>
<td>(15.89/-8.596)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>10% random measurement noise</td>
<td>16</td>
<td>1.4808 (-0.191)</td>
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<td>4.6697 (-0.128)</td>
<td>1.0207 (-6.940)</td>
<td>(162.35/-5.564)</td>
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<td></td>
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<td>100% unmeasured periodic excitations; 1Hz, 2Hz, 4Hz</td>
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<tr>
<td></td>
<td></td>
<td>10% random measurement noise</td>
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</table>

Table 1. Estimated modal parameters for different noise cases.

5 References


