Transmissibility in coupled structures: 
Identification of the sufficient set of coupling forces.

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Abstract
The objective of this paper is to study the transmissibility concept of MDOF systems, with multiple excitations, for coupled structures. Since the transmissibility relates displacements, no force measurements are needed and the transmissibility matrix can be calculated from measured responses only. From the definition of transmissibility, it has been shown that, under certain conditions, the transmissibility matrix for the main structure is the same as the transmissibility matrix for the coupled structure. These conditions arise from the fact that the effect of the additional structure on the main structure is equivalent to a set of generalized forces applied at the joint coordinates. Since the magnitude of the applied forces does not affect the transmissibility matrix, it remains the same, whatever forces are applied at the joint, as long as they are applied at the same coordinates; so these forces can substitute the additional structure. It will be shown if one considers too few forces the transmissibility matrixes are no longer coincident at those modes where the missing forces play an important role. Even when we do not have the complete set of forces introduced by the additional structure, the set can be sufficient for some of the modes or for the frequency range of interest. This paper develops this idea and a FE-approach will prove to be very helpful in finding the sufficient set of applied forces in the frequency range of interest.

1 Introduction
The transmissibility matrix, as used in this paper, was defined for the first time by Ribeiro[1], relating two sets of responses, named as “known” and “unknown” responses. Maia et al [4-6], sequentially improved the concept, and presented some applications for it. The idea in those papers is to monitor the vibration at certain given co-ordinates, using the transmissibility concept: assuming that on these co-ordinates (the unknown set) no transducers can be placed, the techniques allow for the estimation of those responses from the responses at another set of chosen co-ordinates (the known set). From laboratory tests, the relationship between both sets was obtained and it will be used (through a special transfer matrix) to compute the unknown set from the measured known set. This matrix, named as transmissibility matrix, was shown to be the same for a given set of force (and/or moment) locations, regardless of the magnitude of those forces (and/or moments).

One of the applications of the transmissibility concept is in structural coupling, which is the objective of the present paper. The necessary conditions that must be considered were presented in detail in the work dealing with transmissibility in coupled structures [7]; under these conditions, the transmissibility matrix for the main structure remains unchanged even when an additional structure is attached to the main one. To make this property valid it is necessary to consider a sufficient number of forces applied at the joint, so these forces can substitute the effect of the additional structure. Although it might be argued that a reduced number of considered forces would hamper the results since it would not include information about some modes, it will be shown that, as long as there is enough information regarding the modes included in the frequency range of interest the minimum number of forces can be reduced without deterioration of the results.
2 Transmissibility in the Coupling of Sub-structures

2.1 Theoretical Review

In this paper a reduced theoretical model, in order to see the similarity between the theory and the given numerical example, will be presented. A more general theoretical development can be found in the work [7] about transmissibility in coupled structures.

Let us consider Figure 1. From now on, whenever referring to the coupled structure we shall use CS, to the main structure we shall use MS and to the additional structure AS. The exclusive co-ordinates for the MS are named as \( i \), the co-ordinates for both the MS and AS are named as \( j \).

The receptance matrix relates the dynamic forces, applied to the structure, to the displacements, induced by the action of these forces:

\[
\{X\} = [H] \{F\}
\]

(1)

Depending on the co-ordinates type, the FRF matrix can be partitioned into sub-matrices:

\[
\begin{bmatrix}
H^{(CS)}
\end{bmatrix} =
\begin{bmatrix}
H^{(CS)}_{ii} & H^{(CS)}_{ij} \\
H^{(CS)}_{ji} & H^{(CS)}_{jj}
\end{bmatrix}
\]

(2)

In the above relation, \( H^{(CS)}_{ii} \) represents the FRF matrix of the CS, \( H^{(CS)}_{ij} \) - the one that relates co-ordinates \( i \), \( H^{(CS)}_{ji} \) - the one relating co-ordinates \( i \) and \( j \) … and so on.

The FRF matrix of the MS has similar structure:

\[
\begin{bmatrix}
H^{(MS)}
\end{bmatrix} =
\begin{bmatrix}
H^{(MS)}_{ii} & H^{(MS)}_{ij} \\
H^{(MS)}_{ji} & H^{(MS)}_{jj}
\end{bmatrix}
\]

(3)

The conditions of the displacement compatibility and the force equilibrium imply that:

\[
\{X^{(MS)}_j\} = \{X^{(AS)}_j\} = \{X^{(CS)}_j\}
\]

(4)

\[
\{F^{(CS)}_j\} = \{F^{(MS)}_j\} + \{F^{(AS)}_j\}
\]
Assume that there are two different sub-sets of responses, named $K$ and $U$, respectively meaning known and unknown, for the $MS$.

For the set of co-ordinates of the applied forces, named $A$, we only consider the set of co-ordinates $j$; this set contains the co-ordinates with forces introduced by the $AS$ through the coupling co-ordinates on the $MS$:

\[
\{F_j\} = \{F_A\}
\]  

From the definition of receptance and using the sub-sets of co-ordinates above mentioned, it is possible to relate the displacements to the dynamic forces through the following receptance matrix:

\[
\begin{bmatrix}
X^{(CS)}_K \\
X^{(CS)}_U
\end{bmatrix} =
\begin{bmatrix}
H^{(CS)}_{Kj} \\
H^{(CS)}_{Uj}
\end{bmatrix} \cdot \{F^{(CS)}_j\}
\]  

The known and unknown responses of the system become:

\[
\begin{align*}
\{X^{(CS)}_K\} &= \left[H^{(CS)}_{Kj}\right] \cdot \{F^{(CS)}_j\} = \left[H^{(CS)}_{KA}\right] \cdot \{F^{(CS)}_A\} \\
\{X^{(CS)}_U\} &= \left[H^{(CS)}_{Uj}\right] \cdot \{F^{(CS)}_j\} = \left[H^{(CS)}_{UA}\right] \cdot \{F^{(CS)}_A\}
\end{align*}
\]  

These equations are quite known from previous papers [1-5]. The transmissibility matrix was defined as the relationship between the sub-set of unknown displacements and the sub-set of known displacements:

\[
\{X^{(CS)}_U\} = \left[A^{(CS)} T^{(CS)}_{UK}\right] \cdot \{X^{(CS)}_K\}
\]  

Thus,

\[
A^{(CS)} T^{(CS)}_{UK} = \left[H^{(CS)}_{UA}\right] \cdot \left[H^{(CS)}_{KA}\right]^{-1}
\]  

where the number of $K$ co-ordinates must be at least equal to the number of $A$ forces, for the inverse to exist. Note that the $A$ set of co-ordinates need not be distinct from the $K$ set of co-ordinates.

If the $AS$ is removed, the transmissibility of the $MS$ remains unchanged if the applied forces introduced by the $AS$ to the $MS$ are replaced by a substitute set of known forces (and/or moments), using the same coupling co-ordinates $j$:

\[
\begin{align*}
\{X^{(MS)}_U\} &= \left[A^{(MS)} T^{(MS)}_{UK}\right] \cdot \{X^{(MS)}_K\} \\
A^{(MS)} T^{(MS)}_{UK} &= \left[H^{(MS)}_{UA}\right] \cdot \left[H^{(MS)}_{KA}\right]^{-1}
\end{align*}
\]  

As the transmissibility matrix does not depend on the magnitude of the applied forces, as long as these forces are applied at the same set of co-ordinates, then:

\[
A^{(MS)} T^{(MS)}_{UK} = A^{(CS)} T^{(CS)}_{UK}
\]
2.2 Considerations

Some of the forces considered when the set of co-ordinates with applied forces is defined may be zero. An important consequence is that, if one is unsure about considering in the analysis a certain co-ordinate as having an applied force when the structure is coupled, one may include it. The only consequence is the need to measure the response at another co-ordinate and, correspondingly, the matrices involved will increase in size. Using a set of applied forces that contains too many elements can increase the size of the matrix undesirably leading to numerical errors and/or experimental difficulties.

3 Numerical Tests

In order to validate the presented theory, a simple FE model of an airplane was built (Figure 4, 5). Like in the theoretical model, there is an AS, small beam-element, connected to the MS through a joint. The author has chosen a very simple FE-model with the number of elements used just enough to assure a good qualitative simulation of the objective of this paper.

![Figure 4: Main Structure](image1)
![Figure 5: Coupled Structure](image2)

If we look to the real structure we cannot speak of one single coupling point. In reality we can have a lot of coupling points which have a big number of degrees of freedom. However in the analytical model one single point was used, this in order of simplicity without loses of generality of the used method. This reduces the number of degrees of freedom in the coupling point to six. Theoretically we can conclude that in case of one single coupling point the ‘complete’ set of linear independent forces that is needed in order to have the same transmissibility matrix for both structures, coupled and uncoupled is given by the set of generalized forces that excite those 6 DOF (3 rotations and 3 translations). So the complete set of generalized forces needed to be considered in the coupling point would be 3 linear independent moments and 3 linear independent forces, giving us a set of 6 coupling co-ordinates. Thus a minimum of 6 known responses would be needed in order to have a solution for the transmissibility matrix. However, not all of these forces will be introduced by the additional structure, thus the ‘sufficient’ set of linear independent forces can be less. It is our goal to find those forces that must be applied at the coupling co-ordinates, so that these forces can substitute the additional structure at the studied modes. On will see that the forces introduced by the additional structure are depending a lot on the studied modes, thus on the frequency range of interest. If too less coupling co-ordinates are considered the transmissibility matrix will no longer be coincident at those modes were the neglected coupling co-ordinates are important and thus where the added structure is introducing forces that were not taken in account.

The coincidence of the Transmissibility matrixes can be seen as an indicator for the validity of the assumption about the number and co-ordinates of the applied forces introduced by the additional structure on the main structure.

It can be formulated as obvious that the problem of choosing the minimum but sufficient set of applied forces is inherent connected to the choice of the set of known responses, because by changing it, the problem is reformulated and new information about some modes can be introduced, hence this can result in the need to new coupling co-ordinates, applied forces.
The simplicity and the advantages of using a Finite Element Method (FE) will become obvious in the next presented results. After several simulations [7] good conditions were set for the calculation of the transmissibility matrices, valid for both the main structure and the coupled structure. The set of three applied forces (F1, M1 and M2) and the set of 3 known responses (K1, K2 and K4) gave very good results and can validate the presented theory.

Figure 6: Responses                              Figure 7: Set of generalized forces

The results of this setup are shown in the graphics below (Fig. 8a-c).

Figure 8a, 8b, and 8c: Elements of the Transmissibility matrix of the CS and MS

What we see is a good coincidence in the biggest part of the studied frequency range. A small deviation can be noticed in the frequency range of 75-100Hz. In order to understand the quality of these results, we had a closer look at some of the structural modes of this airplane structure. The methodology followed will be able to give us some very important information, we can not only confirm previous made results but it will help us to continue and improve our results and shown the meaning of the “sufficient” set of coupling forces. In particular I wanted to have a close look at three modes, a mode at the left side of this frequency range were we found some deviations, a mode at right side of this frequency range and a mode in the problematic frequency range. The objective was trying to really understand what’s happening at the joint co-ordinates; more specifically which relative movements between both ends of the joint are induced by the AS. If we know the relative movements that are taking place between the two ends of the joint, we can easily decide which set of generalized forces may be used to simulate the effect of the AS. The need of force F1 will be considered as obvious, taking in account the gravitational force of the AS applied on the MS. (If F1 is not used an amplitudes shift at almost all modes was found [7])

A modal analysis was done in order to have the results of the nodal displacements, ROTX (rotation around x-axis), ROTY (rotation around y-axis) and ROTZ (rotation around z-axis). This was respectively done for the modes at 17.74 Hz, at 82.389Hz and 154.757Hz. We have a normal Cartesian co-ordinate system with the X-axis in the direction of the body of the plane and the Z-axis in the same direction as the tail perpendicular to the wings.
MODE 1: 17.74Hz

Figure 9a: ROTX  
Figure 9b: ROTY  
Figure 9c: ROTZ

MODE 2: 82.389Hz

Figure 10a: ROTX  
Figure 10b: ROTY  
Figure 10c: ROTZ

MODE 2: 154.757Hz

Figure 11a: ROTX  
Figure 11b: ROTY  
Figure 11c: ROTZ

These pictures are making it the reader of this paper easier to understand better the different modes that were investigated. One has to look in particular to the relative movement of the AS with respect to the support structure on the wing. In this way a qualitative study of each movement can be done. But in order to have a good quantitative study of the importance of each rotation in each of the three modes, I will investigate the relative rotation of node 1 with respect to node 2 (Figure 12). The importance of each rotation in each mode will be displayed in a percentage (Table 1).

Figure 12: Detail of AS, joint and support structure with 2 nodes drawn in red

<table>
<thead>
<tr>
<th>Mode 1: 17.740Hz</th>
<th>RotX</th>
<th>RotY</th>
<th>RotZ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Node 1:</td>
<td>6.9926</td>
<td>0.58540E-01</td>
<td>0.24367E-05</td>
</tr>
<tr>
<td>Node 2:</td>
<td>7.0088</td>
<td>0.58783E-01</td>
<td>0.24444E-05</td>
</tr>
<tr>
<td>Relative Movement:</td>
<td>0.0.162</td>
<td>0.000243</td>
<td>0.774E-08</td>
</tr>
<tr>
<td>Importance of Movement:</td>
<td>98.5 %</td>
<td>1.48 %</td>
<td>0.02%</td>
</tr>
</tbody>
</table>
Looking at the first mode, we see that the rotation ROTX (98.5%) can be considered as almost the only relative rotation of the AS with respect to the support structure on the wing at this mode. You can conclude that moment M2 (this moment induces this rotation) must play an important role at this mode. This moment is almost completely simulating the effect of the AS on the MS at this mode. This is why we can see a perfect coincidence of the transmissibility matrices in the vicinity of the frequency of this mode. Even if we would only use the moment M2 and force F1 and we calculate the transmissibility matrices (with U1, K1 and K2) for main and coupled structure we can find a perfect coincidence in the vicinity of the frequency of 17.74 Hz.

![Figure 13a and 13b: Elements of the Transmissibility matrix of the CS and MS](image)

Beside the confirmation of the coincidence in the vicinity of the considered mode, some other conclusions can be taken when studying these curves. Both elements of the transmissibility matrix showing the same evolution of coincidence until 140Hz, and a bad coincidence from further on, concluding that there some forces must be missing. We also see that in the vicinity of 154.76Hz we see a difference in element T12 that is not observable in element T11. This already gives us a hint that the co-ordinates of the known response also play an important role in the coincidence of both transmissibility matrices. We already know from above results that the moment M1 plays an important role at this mode (154.76Hz, ROTY 85.9%). This moment is not considered in the above
setup, explaining why we do not have a coincidence in the curves at this mode. The FEM is able to give us the answer to this difference is only seen in element T12 and not in element T11. It seems that co-ordinate K1 is exactly placed on a node (point of zero-displacement) of this mode and co-ordinate K2 on a point of extreme displacement, leading to following generalized conclusion about the sufficient set of forces and the appropriate choice of known co-ordinates:

“For every different set of known responses the problem will be different and other forces may be needed and some can be eliminated. To really understand this you must think that in every mode the substructure will introduce different generalized forces, coupling co-ordinates, on the MS. Let’s consider that for one of the modes one force is introduced that is not introduced by the other modes. If now happens that we are doing our measurement exactly in the nodes of this mode, and thus having no information about this mode, the force will not be needed in the sufficient set of generalized forces to obtain the coincidence of both transmissibility matrices. From the moment we have information about a mode, from which we did not take in account the forces the AS induces on the MS, the transmissibility matrices will no longer be coincident in the vicinity of the frequency of that mode.”

Using the setup with the generalized set of forces (F1, M1 and M2) we can see that in the vicinity of the modes 1 and 3 the coincidence of the transmissibility matrix is perfect. This is clearly understood considering above results that the two moments M1 and M2 are able to induce those rotations that are playing an active role at these modes.

If we now have a look at the mode at 82.389Hz we can see a significant importance (22.83%) of the rotation ROTZ that was not seen in the other investigated modes (0.02% and 0.22%). This lets us conclude that a new moment, that induces this rotation at this mode, must be considered in the minimum but sufficient set of coupled co-ordinates. Doing this and choosing an appropriate set of known responses probably will ensure the coincidence of both transmissibility matrices in the vicinity of this frequency. In the next setup a set of 4 applied generalized forces (figure 16) at the end of the joint will be applied in a try to completely simulate the effect of the AS on the MS in the entire studied frequency range. M3 is the new considered moment and K5 (figure 15) the extra known response.

The numerical results found for this setup are given in the next four graphics (Fig 17a-d):
These results are completely satisfying the above expectations; we see a perfect coincidence of both transmissibility matrix among the entire studied frequency range. One now have to consider if it is worth to increase the number of coupling co-ordinates by one to have this small improvement and complicate both numerical and experimental tests in a significant way (in the experimental tests this improvement was even less [7]).

4 Conclusions

It was shown that, under certain conditions, the transmissibility matrix of the main structure remains unchanged, even if an additional structure is coupled to the main one. The methodology followed in the numerical simulation, concerning the relative movements between both ends of the joint element, seemed to be a good indicator for the importance of each force introduced by the additional structure. It showed again that for each mode the minimum set of applied forces can be different, although the sufficient set of applied forces in our study must contain all forces that play a significant role in at least one of the modes in the frequency range of interest. In a more complex situation with several joints, this method could be useful, hence not at every joint element the same set of forces must be considered. If no numerical model is available, one could think of using strain-transducers, attached to the joints in order to discover the sufficient set of applied forces.

5 Acknowledgements

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6 References


