Avoiding modal aliasing when analyzing responses with partial force measurement

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Abstract

Formulating modal identification algorithms in continuous time, assuming band limited measurements prevents modes from outside the data bandwidth from aliasing into the analysis. This paper details how this technique may be coupled with partially measured force inputs to provide for an efficient generalized operational modal analysis algorithm.

1 Introduction

Experimentalists are familiar with the aliasing that happens in data acquisition when the sampling rate is less than twice the highest frequency of energy in the signal to be sampled. Much effort has been made over the years using a combination of analog and digital filters to make sure that the higher frequencies are filtered out to avoid or minimize the effect of this aliasing. Much less talked about is the aliasing that occurs in modal parameter estimation, or curvefitting, when the residual effects of out of band modes violate the assumptions of the finite dimensional parametric model that the experimentalist uses to curvefit the acquired digitized data. While the out of band energy has been filtered out of the now band limited data, the tails, sometimes called residual flexibility and inertial restraint of the out of band modes are still present in the data.

Much has been written about relative virtues of frequency domain and time domain modal parameter estimation methods, but we claim that the real difference is between continuous time and discrete time, or equivalently, between the Laplace domain and the z domain. The Laplace domain has an infinite frequency range, whereas the z domain describes a finite frequency band, with out of band frequencies being aliased back into the finite range. There is in fact a many to one mapping from the Laplace domain into the z domain given by

\[ \lambda \in \mathbb{C} \rightarrow \exp(\lambda t), \text{ where } \Delta t \text{ is the discrete time step.} \quad (1) \]

We see that the imaginary axis in the Laplace domain gets wrapped around the unit circle in the z domain, and that for any integer \( k \),

\[ \lambda + \frac{2\pi ik}{\Delta t} \rightarrow \exp(\lambda \Delta t + 2\pi ik) \equiv \exp(\lambda \Delta t). \quad (2) \]

If we now inspect Figure 1, where we have selected a frequency band of interest, it is clear that the residual effects of the modes outside the band are still present even if the data is bandpass filtered.
A modal parameter estimation algorithm formulated in continuous time, or equivalently, the Laplace domain, will in many cases be able to estimate modes outside the band if the residual effects are rise above the noise floor. Any method formulated in discrete time, or equivalently, the \( z \) domain, will suffer the curse of equation (2), so that poles corresponding to out of band modes will be mapped into the analysis band. This phenomenon is called modal aliasing which gives rise to non-physical or computational modes. All wideband modal parameter estimation methods give rise to some computational modes, but the \( z \) domain (read discrete time) methods will guarantee that the computational modes corresponding to out of band modes will be mapped back into the analysis band and mingle with the physical modes.

Algorithms for performing the triage of candidate poles are hence important for both the discrete and the continuous time formulations, but the job is easier when out of band residual effects do not cause aliasing.

1.1 Aliasing as classification of widely used methods

1.1.1 Aliased – \( z \) Domain and Discrete Time

1. Least Squares Complex Exponential (Polyreference)
2. Least Squares Complex Frequency Domain (PolyMax)
3. Ibrahim Time Domain and ERA

2.1.1 Alias Free – Continuous Time and Laplace Domain

1. Rational Fraction Orthogonal Polynomials
2. Simultaneous Frequency Domain (Coppolino, Link, Vollan)
2 AFPoly, the Alias Free Polyreference Method

This method is based on the Laplace domain orthogonal polynomial rational fraction method, with the added usage of a generalized orthogonal polynomial companion matrix, which allows for the numerical stability over an entire frequency range containing hundreds, even thousands of modes.

For an example of an orthogonal polynomial, see Figure 2.

![Graph of Orthogonal Polynomial Basis Function](image)

**Figure 2 Orthogonal polynomial**

It does not really matter which system of orthogonal polynomials we use, as long as we solve the characteristic equation in terms of the orthogonal polynomials as basis functions, for example by using the generalized companion matrix eigensolution. The drastic difference in numerical precision when using the generalized companion matrix may be see from Figure 3 which compares this method to using a power polynomial basis. The point is that using orthogonal polynomials also during the root solve gives us the same benign numerical properties as the z domain methods, but with the problem of modal aliasing.
Figure 3 Numerical error in polynomial root solving using power polynomials and orthogonal polynomials as basis functions

The key observation is that we are working on test data which are bounded in the frequency domain and that hence, appealing to the sampling theorem, the formulation may be phrased in terms of continuous time, or, equivalently, the Laplace domain.

Now, the orthogonality properties of the orthogonal polynomials in the Laplace domain induces an orthogonality property of the inverse infinite Fourier transform of these polynomials in the continuous time domain. Since the orthogonal polynomials are nonzero only over the bounded frequency range of the bandpass filtered test data, the inverse infinite Fourier transform of these polynomials, the so called Green’s functions are well defined wavelets with finite energy, see Figure 4. It is a remarkable fact that when we use the Legendre polynomials as a basis, the corresponding Green's functions are half order Bessel functions.
Hence, in the continuous time domain, the usual differential equations of motion translate into convolution integrals with these finite energy Greens functions, and the usual numerical problems with higher order differentials vanish. Using these convolution integrals, we can also annihilate the differential operators on the force contributions in the time domain, such that meaningful instrumental variables may be defined in order to process response data with unmeasured broadband excitation. The size of the equation systems is minimized by considering the adjoint system, exchanging the role of excitation and response, just as is done in the Polyreference method.

The principal advantage of this method is that the residual effects of modes outside the frequency band of analysis are kept outside this band, and hence, the nonlinear effects of aliasing are avoided. One may compare the improved clarity of an AFPoly stability diagram compared to the aliased complex exponential diagram in Figure 5.
3 Partially measured forces in the AFPoly method

We first formulate the basic equations in the time domain as

$$\sum_{k=0}^{N} A(k) \frac{d^{N-k} Y(t)}{dt^{N-k}} = \sum_{k=0}^{M} B(k) \frac{d^{M-k} F(t)}{dt^{M-k}} + \sum_{k=0}^{P} C(k) \frac{d^{P-k} N(t)}{dt^{P-k}} N(t)$$

(3)

where $Y(t)$: response, $F(t)$: measured force, $N(t)$: unmeasured force.

Define the correlation $\Gamma(R, P) = \int \left( \sum_{k=0}^{P} C(k) \frac{d^{P-k} N(t)}{dt^{P-k}} N(t) \right)^R Q_R \left( \frac{d}{dt} \right) \left\{ \frac{Y(t)}{F(t)} \right\} dt$,

(4)

where $Q_R (\bullet)$ is the orthogonal polynomial of order $R$. We have $\Gamma(R, P) = 0$ when $R > P$ and $N(t)$ is band limited and has a sufficiently smooth spectrum. The annihilation of the unmeasured force term in the correlation is due to the fact that the polynomial $Q_R$ is orthogonal to all polynomials of lower order.

The term sufficiently smooth means in particular that $N(t)$ must have a spectral density, sine waves need not apply, but we need not demand white noise either. It may be prudent to apply a mollifier window in the time domain to soften the effect of deterministic components, noting that when such a window is used, the time histories are no longer stationary, and time and state averages are not the same, so that care must be taken in computing correlation estimates.

Since the unmeasured forces are cancelled in the higher lag correlations as defined in (4), the remaining terms in the system equations (3) are linear in the unknowns. The reader may spot this as
another instrumental variable approach. We can hence solve for the characteristic equation in orthogonal polynomial coordinates, and enjoy a numerically stable alias free solution.

4 Conclusion

This paper has defined the concept of modal aliasing, and shown how its damaging effects may be avoided by using a continuous time formulation. In particular, we highlighted the alias free Polyreference method which uses an orthogonal polynomial basis is the Laplace domain, and which can be used to cancel out the moving average terms in the unmeasured excitations.

5 References

