On Modal Kinetic Energy and Effective Independence

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Abstract

The comparison and inherent relationship between two influencing sensor placement methods, i.e. Modal Kinetic Energy and Effective Independence, are addressed in this paper. The problem is of primary concern for dynamic testing, damage identification and structural health monitoring. By analyzing the sensor placement problem with Effective Independence method from the perspective of a new reduced system, the connection of Modal Kinetic Energy with Effective Independence method is revealed. The latter is an iterated version of the former, and the reduced mode shapes are ortho-normalized repeatedly during iterations of the latter. Two alternative forms for efficient computation of the iterative Effective Independence method are presented. Finally, both methods are applied to the I-40 Bridge, and the relationship is verified.

1 Introduction

Before outlining the paper, we describe the motivations, introduce the basics of the two sensor placement methods, and discuss their differences and connections.

1.1 Motivations

The problem of sensor placement is an importance issue in dynamic testing of large structures, and has been investigated from different approaches, as can be seen from the abundance of literature [1-3], and the references therein. Especially due to the increasing interest of damage identification and structural health monitoring (SHM) in last two decades, more researchers are involved in this topic, and methods from various perspectives are proposed [4-5]. The key ideas behind these approaches are, however, similar. Most sensor placement methods aim to achieve best sensitivity changes’ detection of signatures indicating damage, or the best identification of structural characteristics, including the eigen-frequencies, mode shapes, and also damping ratios. Although a single eigenfrequency or mode shape is not definitely sensitive and sufficient enough for detecting the existence of damage in SHM, the combination of them, such as those used in subspace based damage identification method [6], shows great potentials. In this connection, the eigenfrequencies and the mode shapes are to be identified as accurately as possible, and server as a benchmark database for the following model updating, and damage identification. This is the objective of the two methods to be discussed, the Modal Kinetic Energy (MKE) and Effective Independence (EI) method, which is the most influential and commonly used. The EI is recommend by Ewins[7] and Friswell etc. [8-9] for sensor placement, and already embedded in commercial software MSC/NASTRAN[10]. Although the theory of both MKE and EI methods are quite straightforward, and well developed, and both are widely discussed and applied, the same degree of understanding cannot be said to exist. Researchers may notice that EI can arrive at similar results as that of MKE in many
circumstances, and may have a vague feeling that MKE and EI have something in common. Their relationship, which was thought to be important on the basis of theoretical considerations and on the development of other effective sensor placement methods, is not explicitly and mathematically reported. This is the topic of our present work. From the viewpoint of the authors, the difference and consistency of MKE and EI will increase the understanding of both methods, and the role of each candidate sensor position played in EI.

1.2 Problem formulation of sensor placement

The sensor placement problem can be investigated from uncoupled modal coordinates of governing structural equations as follows,

\[
\begin{align*}
    \ddot{q}_i + M_i^{-1}C_i \dot{q}_i + M_i^{-1}K_i q_i &= M_i^{-1}\Phi^T B_0 u, \\
    y &= \Phi q + \varepsilon
\end{align*}
\]  

where, \( q_i \) is the \( i \)th modal coordinate, \( M_i \), \( K_i \) and \( C_i \) are the corresponding \( i \)th modal mass, stiffness and damping matrix, respectively, \( \Phi \) is the mode shape matrix with its \( i \)th column as the \( i \)th mass-normalized mode shape, \( B_0 \) is simply a location matrix formed by ones (corresponding to actuators) and zeros (no loadings), specifying the positions of the force vector \( u \). \( y \) is a measurement column vector, indicating which positions of the structure are measured, and \( \varepsilon \) is stationary Gaussian white noise with variance of \( \psi_\varepsilon^2 \).

Sensor placement problem described in Model (1) is, essentially, divided into three aspects. Firstly, what is the least number of accelerometers required to be installed in a structure for a successful dynamic testing? Secondly, where should these accelerometers be installed, including those additional ones if available? And if these additional sensors are available, should we install them as redundant sensors or placement them in other positions? Lastly, how could we evaluate the effectiveness of different sensor placement methods.

The first problem is partly resolved. It is known that the minimum number of sensors to be instrumented could not be less than the number of mode shapes to be identified, which is determined by the observability of the system. Moreover, the practical number of sensors, which is limitedly preset before test due to the availability of instruments, is usually larger than the minimum number because of the requirement of visualization of the mode shapes [11].

The second problem is the core and amazing one, which depends largely, however, on the third aspect. As mention above, the MKE and EI methods discussed in this paper aim both to the best identification of the eigen-frequencies and mode shapes. Then, only the second problem is left. Without loss of generality, it is assumed here that the total degree of freedoms (DOFs) of the structure described in Model (1) is \( n \), the number of mode shapes used for sensor placement is \( k \), and the available number of sensors is \( m \). Then, the sensor placement problem becomes, basically, where to deploy the \( m \) available sensors out of the total \( n \) DOFs of a structure for dynamic testing, \( i.e., \), which rows of the measurement vector \( y \) in Eq.1 are to be selected. MKE and EI give apparent different solutions to this problem. However, there is an underlying connection between these two solutions, which is unknown before and will be discussed in this paper.

1.3 Outline of the paper

This paper is structured as follows. Section 2 describes the rationale and basis of the MKE and EI methods. In Section 3, the connection of both methods is derived, and the physical significance of
the EI method is recapitulated. Section 4 discusses the effect of non homo-generous mass distribution on MKE and computation aspects of EI. And in Section 5, both MKE and EI are applied on the I-40 Bridge to verify their connection. Finally, the result and contribution of this paper is presented.

2 Modal Kinetic Energy and Effective Independence

Both MKE and EI methods have found many applications in actual dynamic testing, and obtained reasonable results. Their theoretical background and rationale are to be explained in this section. The material presented here is well known [1-3], and expounded repeatedly just for the sake of clarity and our derivation of their relationship in Section 3.

2.1 The Modal Kinetic Energy method

The MKE provides a rough measure of the dynamic contribution of each candidate sensor to the target mode shapes. The reason to adopt MKE resides in that it tells which DOFs captures most of the relevant dynamic features of the structure. MKE helps to select those sensor positions with possible large amplitudes and to increase the signal to noise ratio, which is critical in harsh and noisy circumstances [2,3].

The method ranks all candidate sensor positions by their MKE as follows,

\[ MKE_{pq} = \Phi_{pq} \sum_s M_{ps} \Phi_{sq} \]  

where \( MKE_{pq} \) is the kinetic energy associated with the \( p \)th degree of freedom in the \( q \)th target mode, \( \Phi_{pq} \) is the \( p \)th component in the corresponding \( q \)th mode shape, \( M_{pq} \) is the term in the \( p \)th row and \( s \)th column of the mass matrix \( M \), and \( \Phi_{sq} \) is the \( s \)th coefficient in the \( q \)th target mode shape. The sensor locations with higher values of MKE are selected as measurement sensor set.

2.2 The Effective Independence (EI) method

The aim of the method is to select measurement positions that make the mode shapes of interest as linearly independent as possible while containing sufficient information about the target modal responses in the measurements [2,12-14]. The method originates from estimation theory by sensitivity analysis of the parameters to be estimated, and then it arrives at the maximization of a norm of the Fisher information matrix, for instance, the determinant or the trace, as well as to minimize the condition number of the information matrix. It is reflected in the coefficient variance-covariance matrix. Thus, the covariance matrix of the estimate error of the modal coordinates would be minimized. The number of sensors is reduced in an iterative fashion from an initial candidate set by removing sensors from those degrees of freedom, which contribute least among all the candidate sensors to the linear independence of the target modes. Finally, it preserves the required necessary candidate sensors as the optimal sensor set.

From the measurement output expression in Eq.1, the EI analyzes the covariance matrix of the estimate error for an efficient unbiased estimator as follows,

\[
E[(q - \hat{q})(q - \hat{q})^T] = \left( \frac{\partial y}{\partial q} \right)^T \Omega_0 \left( \frac{\partial y}{\partial q} \right)^{-1} = \left( \frac{1}{\Omega_0^2} \Phi^T \Phi \right)^{-1} = Q^{-1}
\]  

(3)
in which \( Q \) is the Fisher information matrix. Maximizing \( Q \) will result in the best state estimate. In practice, the analysis begins by solving the following eigen-value equation,

\[
[\Phi^T \Phi - \lambda I] \Psi = 0
\]

where \( \Psi \) are the orthogonal eigenvectors. The effective independence coefficients of the candidate sensors are computed by the following formation,

\[
E_D = [\Phi \Psi] \otimes [\Phi \Psi]^{-1} \{1\}_n
\]

in which \( \otimes \) represents a term-by-term matrix multiplication, \( \{1\}_n \) is an \( n \times 1 \) column vector with 1 elements. \( E_D \) is the EI indices, which evaluates the contribution of a candidate sensor location to the linear independence of the modal partitions \( \Phi \).

The selection procedure is to sort the elements of the \( E_D \) coefficients, and to remove the smallest one at a time. The \( E_D \) coefficients are then updated according to the new modal shape matrix, and the process is repeated iteratively until the number of the remained sensors equals to a preset value. The remained DOFs serve as the measurement locations, referring to [2] for details.

3 The relationship between modal kinetic energy and effective independence methods

In previous investigations of comparisons among different sensor placement methods [2,3,12-15], the EI and MKE show similar results, especially for the first several iterations of the EI and MKE in the cases of structures with homogeneous mass distributions. However, the connection between both influential methods is not clearly understood, at least to the knowledge of the authors from the literature. In this section, the latent connection between both methods is revealed.

For simplicity to expose the relationship between MKE and EI, an identity mass matrix is assumed at first, and then the effects of non-identity equivalent mass matrix on sensor placement will be discussed in Section 4. Under the assumption of an identity mass matrix and normalized mode shapes, the MKE index can be rewritten as the following formula,

\[
MKE = diag(\Phi \Phi^T)
\]

where, operator \( diag \) denotes a column vector formed by the diagonal terms of a matrix. Similarly, the EI index can be alternatively computed as the diagonal of the following matrix [2],

\[
E_D = diag([\Phi^T \Phi]^{-1} \Phi^T)
\]

From the above Eq.6 and Eq.7, it can be certainly observed that the result of the first iteration of EI should be the same as that of the MKE, which is already shown clearly in the examples of references using EI methods [2,3,14-15]. This is simply due to the fact that the mode shape matrix is normalized in the first iteration of EI, and the middle term in the right hand side of Eq.7 is just an identity matrix. The two formations of EI and MKE are identical under this circumstance. It is, therefore, unnecessary to apply MKE first when implementing EI as originally proposed by Kammer[2]. In the following iterations, the EI indices are weighted by a term of the reduced Fisher information matrix, but the MKE is not. And this is why EI is different from MKE afterwards.
Moreover, if the EI in the second iteration is considered, we found that the measured sensor output formulated in Eq.1 should be rewritten because a previously assumed output component is not measured anymore. Without loss of generality, the $k^{th}$ index of EI in Eq.5 in the first iteration is assumed to be the smallest and to be excluded. Then, the reduced output vector in Eq.1 should be reformulated as follows,

$$y_1 = \Phi_1 q_1 + \varepsilon_1$$

where, $y_1$ denotes the remained measurements with the $k^{th}$ measurement deleted in $y$ of the Eq.1, and likewise, $\Phi_1$ is the same mode shape matrix as $\Phi$ in Eq.1 only with the $k^{th}$ row deleted. The model described in Eq.8 becomes a reduced system with only $n-1$ DOFs since the previous $k^{th}$ DOF in the original model is rejected.

Basically, we view sensor placement broadly as a problem of system reduction, and the low-dimensional reduced system defined in Eq.8 is to represent the original full-scale system in Eq.1 as exactly as possible. The information discarded by excluding $n-k$ sensor positions should be insignificant compared to the $k$ sensors retained.

In this new reduced structure with order of $n-1$, the mode shape matrix should be renormalized as the original system. Following the same procedures similar through Eq.3 to Eq.5 with ortho-normalized mode shapes ($\Phi_1^T \Phi_1 = I$), a formulation with the same rationale can be easily obtained,

$$E_{D_1} = \text{diag}(\Phi_1 [\Phi_1^T \Phi_1]^{-1} \Phi_1^T) = \text{diag}(\Phi_1 \Phi_1^T)$$

The EI index in Eq.9 is degenerated once again in form into the MKE index of Eq.6 in its 2nd iteration. Therefore, the key difference between EI and MKE is that in the following iterations of EI, the reduced mode shape matrix is not renormalized, but the MKE is initially using an already normalized mode shape matrix. A reorthonormalized EI in its iterations is just MKE.

To strengthen our arguments further, we consider a special case, in which only one mode shape is considered to compute for sensor placement by MKE and EI, respectively. In this case, the MKE indices are just the squares of the mode shape components corresponding to the sensor positions in Eq.6, i.e. $MKE_j = \Phi_j^2$, and the EI indices ($E_{D_1} = \Phi_j^2 / (\sum_{i=1}^{n} \Phi_i^2$) are the squares of the mode shape components only divided a constant (the squared Euclidean norm of the mode shape according to Eq.7). The only difference between both indices is a constant coefficient. Their ranked sequence is the same.

We can now consider EI from another viewpoint. The mode shapes used in EI, regardless of ortho-normalized or not, can be decomposed using orthogonal-triangular decomposition (QR) as follows [16-18],

$$\Phi = QR$$

where $Q$ is an $n \times k$ unitary matrix with the same dimensions as $\Phi$, and $R$ is a $k \times k$ upper triangular matrix. Thus, the EI index can be also computed using the above-decomposed $Q$ and $R$ matrix by substituting Eq.10 into Eq.7,
The expression of EI in Eq.11 is the same in form as that of MKE in Eq.6. In each iteration of EI, ii computes “MKE index” using the reduced ortho-normalized mode shapes, retains those DOFs with large MKEs, and deletes those with small MKEs.

The rationale behind the QR decomposition in Eq.10 is the same as the above reasoning for the idea of viewing sensor placement as system reduction. QR decomposition is, in fact, an extension of the Gram-Schmidt orthogonalization applying to the dependent columns of reduced mode shapes \( \Phi_1 \), which is not strictly orthogonal anymore after some row of the previous orthogonal mode shapes is deleted in the proceeding iteration. Consequently, the QR decomposition in Eq.10 extracts an orthogonal subspace spanned by the columns of \( Q \). The \( Q \) in Eq.10 is an \( n \times k \) ortho-normal matrix. This means that the columns of the reduced mode shape matrix \( \Phi \), resulted from iterations of EI will be remapped onto the subspace spanned by the ortho-normalized columns of \( Q \). And it is just these columns of \( Q \) that will combine to form the reduced measurement vector \( y \). In dynamic testing, it is also these columns of \( Q \) that are identified as mode shapes of the reduced system being measured.

As a result, the difference between MKE and EI is clear. EI requires iteration computations, but MKE not. In the following iterations of EI, it redistributes all the modal kinetic energy into the retaining DOFs and recomputed their MKE index for the reduced system using re-orthonormalized mode shapes. EI is an iterated version of MKE.

### 4 Mass distribution Effects

For cases of non-identity equivalent mass matrix, the above reasoning can be generalized. The MKE index is computed by,

\[
MKE = \text{diag}(\text{M}\Phi\Phi^T) = \text{diag}(\text{M}^{1/2}\Phi\Phi^T\text{M}^{1/2})
\]

where, \( \text{M}^{1/2} \) is the square root of the semi-definite mass matrix \( \text{M} \). In MKE, each candidate DOF is weighted by the corresponding component in the mass matrix. For those DOFs associated with large components in the mass matrix, they are given more weights in the ranking of their importance for sensor placement. Hence, MKE reflects the characteristics of mass distribution for a given structure.

Moreover, the EI index can still be computed using Eq.7 regardless of mass distribution. To analyze the iterations of EI, the Sherman-Morrision-formula can be employed [16].

\[
\left[\Phi_{(i)}^T \Phi_{(i)}\right]^{-1} = \left[\Phi^T \Phi - \Phi_i \Phi_i^T\right]^{-1} = \left[\Phi^T \Phi\right]^{-1} + \frac{\left[\Phi_i \Phi_i^T\right]^{-1}}{1 - \Phi_i^T \left[\Phi^T \Phi\right]^{-1} \Phi_i}
\]

where, \( \Phi_{(i)} \) is the same mode shape matrix as \( \Phi \) only with the \( i^{th} \) row deleted, \( \Phi_i \) is a column vector that is the transpose of the \( i^{th} \) row of the mode shape matrix \( \Phi \) to be deleted in an iteration of EI. The EI index can be computed using Eq.13 to facilitate the iterative inversion of the middle term in Eq.7 in the following iterations.
Since the mode shapes in the original system is ortho-normalized, Eq.13 in the first iteration of EI is reduced to,

\[
\begin{bmatrix}
\Phi_i^T \\
\Phi_i
\end{bmatrix}^{-1} = I + \frac{\Phi_i^T \Phi_i}{1 - \Phi_i^T \Phi_i}
\] (14)

It is obvious that the left hand side of Eq.14 does not deviate much from an identity matrix because the EI method select a row with a smallest norm in the original mode shape matrix to delete. The second term in the right hand side of Eq.14 is just a small perturbation matrix and insignificant. At least the inverse in Euq.14 is a diagonal dominant matrix. Hence, the inverse middle term in Eq.7 can be regarded as weights for the mode shapes of a reduced system after some DOFs have been sequentially deleted. Therefore, the EI method adds different weights for the remaining mode shapes during its iterations, whereas MKE adds weights for each candidate DOF.

Pre-multiplying both sides of Eq.13 with \( \Phi_r \) and post-multiplying with \( \Phi_r^T \), we can obtain a simple efficient expression for the iterative computation of the EI index,

\[
h_{rr(i)} = h_{rr} + \frac{h_{ri}^2}{1 - h_{ii}}
\] (15)

where, \( h_{rr(i)} \) is the \( r^{th} \) diagonal term of the expression in the bracket of the right hand side of Eq.7 with the \( i^{th} \) row of the mode shape matrix as \( \Phi \) deleted, and \( h_{rr} \) is the \( r^{th} \) diagonal term of the full mode shape matrix \( \Phi \), and likewise \( h_{ii} \) and \( h_{ri} \). The \( h_{rr} \) is called as leverage of each predicted value on its actual measurement \( y \) in statistics [17]. As stated above, the EI method iterates to delete rows corresponding to small \( h_{ii} \). The iterative EI index, the change of the EI index after the \( i^{th} \) row of the previous mode shape matrix is deleted, can be efficiently computed using Eq.15 without the computational burden of matrix inversion.

5 Application to the I-40 Bridge

The application of both MKE and EI is demonstrated using the measurement data of the I-40 Bridge, located over the Rio Grande in Albuquerque, New Mexico. The I-40 Bridge consisted of twin spans made up of a concrete deck supported by two welded-steel plate girders and three steel stringers. The tested section has three spans. The end spans are of equal length, approximately 39.9 m, and the center span is approximately 49.4 m long.
There are totally 13 accelerometers amounted along the length of the bridge, for a total of 26 responses. The shaker consists of a 96.5 kN reaction mass supported by three air springs resting on top of drums filled with sand. The shaker was located on the eastern-most span directly above the south plate girder and midway between the abutment and first pier. Figure 1 shows the shaker and accelerometer locations. Full details of the modal testing of this bridge can be found in Farrar, et al.[19].

The mode shapes extracted from the case of “Test t11tr” are used for the computation of sensor placement of both MKE and EI. In this model, there are totally 26 DOFs (n=26), and 6 identified mode shapes (k=6) available. Two cases are considered. In Case 1, 25 accelerometers will be deployed (m=25). This case corresponds to the first iteration of EI. In Case 2, 6 accelerometers will be amounted (m=6). The number of accelerometers in Case 2 is the minimum number to identify the 6 mode shapes.

In Case 1, both MKE and EI pick sensor position “S1” for exclusion, and their ranking sequences are identical. This agrees with our derivation in Section 4. For Case 2, the ranking sequence of both MKE (Fig.2) and EI (Fig.3) in descending order is S7, N7, N3, S3, N11, and S11. All the middle-span positions are picked up in Case 2. For the I40 Bridge, both MKE and EI give the same results for both Cases. It should be noted that EI does have some difference from MKE in other applications as shown in [2,3].

6 Conclusion

Two influential sensor placement methods, *i.e.* Modal Kinetic Method and Effective Independence method, are discussed and compared, and the connection between the two methods is derived. MKE ranks the relative importance of all candidate DOFs by their kinetic energy with the mass distribution associated with each DOF as weights. EI is an iterated version of MKE regardless of mass distribution of a structure. It is found that the latter is an iterated version of the former for the case of equivalent identity mass matrix of a structure.
The application of MKE and EI on the I-40 Bridge shows that both give identical results for sensor number 25, and 6. This paper provides also two efficient alternative forms to compute the EI index. One is the stable QR method as shown in Eq.11, the other is the iterative row deletion method used in Eq.15.

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8 References