Double Modes Determination of a Gearbox Shaft Using Operational Modal Analysis

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ABSTRACT: Double modes are encountered in the theoretical model of structures as two repeated roots. On the other hand, in experimental modal analysis, double modes are two close modes. A problem sometimes arises in resolving whether there are two modes or only one mode in the measured FRFs. In this paper Operational Modal Analysis (OMA) is applied to determine the double modes of a gearbox shaft. First, the gearbox shaft is modeled using Finite Element Method (FEM) and then it is tested using OMA methods. The results show the existence of double modes both from FEM and OMA presenting that OMA methods are capable to recognize the double modes.

1 INTRODUCTION

Structures with some degree of symmetry such as: circular plates, gears or bladed-discs, usually have two modes with the same natural frequencies. These modes are known as double modes. From experimental point of view, double modes are two close modes and it is difficult to decide on the correct number of modes (one or two). Maia and Ewins (1987) suggested a curve fitting procedure for intelligent analysis of double modes. Afolabi and Alabi (1991) also proposed a method to recognize the double modes using singularity theory. In the present paper, OMA methods are used to model a gearbox shaft. It is shown that the OMA methods are capable to separate and identify the double modes of the gearbox shaft. First the gearbox shaft modelled using Finite Element Method (FEM) showing the shaft had the double modes. Next two OMA methods namely Frequency Domain Decomposition (FDD) in frequency domain and Stochastic Subspace Identification (SSI) in time domain used to identify the modes of gearbox shaft. It is shown that both methods can determine the double modes.

2 THEORY

2.1 Frequency Domain Decomposition (FDD) method

The FDD is a non-parametric Operational Modal Analysis (OMA) method which was first suggested by (Brincker et al 2000). The method is based on one of the modal testing techniques known as CMIF (Ewins 2000). If the system has input $x$ and output $y$, we have:

$$G_y(j\omega) = \overline{H(j\omega)} G_x(j\omega) H^T(j\omega)$$  \hspace{1cm} (1)

where $G_x$ is the Power Spectral Density (PSD) matrix of the input. $G_y$ is the PSD matrix of responses. $\overline{H(j\omega)}$ is the transpose of Frequency Response Function (FRF) matrix, the over bar on $H$ denotes the conjugate of the FRF matrix.

The PSD matrix of the output can be written in the fractional form as (Brincker et al 2000):
\[
G_{yy}(j\omega) = \sum_{k=1}^{n} \left[ \frac{Q_k}{j\omega - \lambda_k} + \frac{\overline{Q}_k}{j\omega - \overline{\lambda}_k} \right] \cdot G_{xx}(j\omega) \left[ \sum_{l=1}^{n} \frac{\overline{Q}_l}{j\omega - \overline{\lambda}_l} + \frac{\overline{Q}_l}{j\omega - \overline{\lambda}_l} \right]
\]  
(2)

where \(Q_k\) is the residue for \(k^{th}\) mode and \(\lambda_k\) is the natural frequency for \(k^{th}\) mode and \(n\) is the number of modes of system.

If the input is assumed to be white noise and the damping is low, Eq. (2) can be written as:

\[
G_{yy}(j\omega) = \sum_{k=1}^{n} \frac{a_k \phi_k \phi_k^H}{j\omega - \lambda_k} + \frac{\overline{a}_k \phi_k \phi_k^H}{j\omega - \overline{\lambda}_k}
\]  
(3)

where \(a_k\) is a scalar, \(\phi_k\) is the \(k^{th}\) mode shape vector.

A mode dominates the dynamic behavior of a structure close to each natural frequency. Therefore the response of structure in this frequency is similar to the mode shape of structure in this mode (Brincker et al. 2000). If the corresponding power spectral density matrix of the response in this frequency is decomposed by taking SVD of the matrix, we have:

\[
\tilde{G}_{yy}(j\omega) = U_j S_j U_j^H = u_{i1} s_{i1} u_{i1}^H + u_{i2} s_{i2} u_{i2}^H + ...
\]  
(4)

where the matrix \(U_j = [u_{i1}, u_{i2}, ..., u_{in}]\) is the unitary matrix including the singular vectors \(u_{ij}\), and \(S_j\) is a diagonal matrix including the singular values \(s_{ij}\).

In the vicinity of the \(k^{th}\) natural frequency, the \(k^{th}\) mode governs the dynamics of structure. Therefore, the first singular value corresponding to \(k^{th}\) natural frequency is in the higher level than the other singular values of that frequency (Brincker et al. 2000). If the structure has the double modes, they dominate the dynamic behavior of structure close to their natural frequency. Thus, the two repeated modes have the singular values in the higher levels than other modes. However, there is a small difference between the levels of singular values of these modes as they are not excited with the same level of force.

### 2.2 Stochastic Subspace Identification (SSI) method

The tradition in time domain based modal parameter estimators come from algorithms based on Correlation Functions. The Stochastic Subspace Identification (SSI) method uses a state space model, a method that convert the \(2^{nd}\) order problem to \(1^{st}\) order problems. Eq. (5) shows a discrete formulation of an output only state space model. \(y_i\) is the output that is generated by the process noise, \(w_i\) and the measurement noise \(v_i\). The dynamics of the physical system is modeled by the state matrix \(A\). The observable part of system dynamics is extracted from the state vector by forward multiplication of the observation matrix \(C\) (Overschee and De Moor 1996).

\[
x_{i+1} = Ax_i + w_i
\]  
(5)

\[
y_i = Cx_i + v_i
\]

where \(x_i\) is the Kalman sequence that is found in the SSI method by an orthogonal projection technique.

\[
\hat{x}_{i+1} = A\hat{x}_i + K_ie_i
\]  
(6)

\[
e_i = y_i - C\hat{x}_i
\]

where \(K_i\) is called the non-steady state Kalman gain and \(e_i\) is called the innovation and is a zero-mean Gaussian white noise process.

The next step is to solve the regression problem for the matrices \(A\) and \(C\), and for the residual sequences \(w_i\) and \(v_i\). Finally in order to complete a full covariance equivalent model in discrete time, the Kalman gain matrix is estimated in Eq. (5). By rearranging Eq. (5), the state space system Eq. (6) is obtained as:

\[
\dot{x}_{i+1} = Ax_i + K_ie_i
\]  
(7)

\[
y_i = C\hat{x}_i + e_i
\]
By performing an eigenvalue decomposition of the matrix $A$ as $A = V [\mu] V^T$ and introducing a new state vector $z_t = V^T x_t$, Eq. (7) can also be written as Eq. (8):

$$z_{t+1} = \left[\mu\right] z_t + \Psi e_t$$

$$y_t = \Phi z_t + e_t$$

where $[\mu]$ is a diagonal matrix holding the discrete poles related to the continuous time poles $\lambda_i$ by $\mu_i = \exp(\lambda_i \Delta t)$, and where the matrix $\Phi$ is holding the left-hand mode shapes and the matrix $\Psi$ is holding the right-hand mode shapes (Brincker et al 2006).

The final results are achieved by a singular value decomposition of the full observation matrix and extracting a subspace holding the modes in the model. This will lead to Unweighted Principle Component (UPC), which is one of the SSI estimation classes.

3 CASE STUDY

3.1 Finite Element model

As a case with a degree of symmetry, a gearbox shaft was considered (Fig. 1). The FEM model of shaft was built using eight node elements of ANSYS software (Fig. 2). Then, the natural frequencies and mode shapes were obtained.

![Figure 1: Gearbox shaft.](image1)

![Figure 2: FEM model of the gearbox shaft.](image2)

Table 1 shows the first three natural frequencies of shaft. The first two modes are close together, presenting the double modes. Fig. 3 illustrates the first and second mode shapes of shaft.

Table 1: FEM natural frequencies.

<table>
<thead>
<tr>
<th>No. of mode</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>FEM natural frequency (Hz)</td>
<td>2480.4</td>
<td>2484.0</td>
<td>6127.2</td>
</tr>
</tbody>
</table>
3.2 Experimental model

The conventional modal analysis was conducted on the shaft using an accelerometer type A123E and a force transducer type 8200 and a charge amplifier type 2647A. A 3560D analyser was used to extract the measured FRFs. The shaft was modelled by exciting the structure at six points in the hammer test as shown in Fig 4.

![Image of test setup for experimental modal analysis]

Figure 4: Test setup for experimental modal analysis.

The frequency range of measurement was 0-3000 Hz. The measured FRFs show only one mode at 2478.223 Hz and the double modes can not be recognized from the results at this frequency (Fig. 5).

![Image of Frequency Response Functions (FRFs) from Modal Analysis]

Figure 5: Frequency Response Functions (FRFs) from Modal Analysis.
3.3 FDD model

The shaft was tested using six accelerometers installed at six points of shaft as shown in Fig. 6.

![Figure 6: Test setup for OMA.](image1)

The shaft was excited at different points by a hammer and the responses were measured. The advantage of OMA methods is that the structure can be excited at different points and different directions and therefore the double modes are excited and can be detected. This is not usually the case in conventional modal testing in which the structure is excited at one point and one direction. The FDD method was applied to obtain the SVD curves as shown in Fig. 7.

![Figure 7: Spectral density of the all data sets from FDD.](image2)

Figure 7: Spectral density of the all data sets from FDD.

![Figure 8: MAC matrix between the first two mode shapes from FDD.](image3)

Figure 8: MAC matrix between the first two mode shapes from FDD.
The double modes are detected from the spectral density of the all data sets from FDD (Fig. 7). This is due to the nature of SVD which can separate the double modes. Fig. 8 shows the comparison of extracted mode shapes of the double modes from FDD.

The two mode shapes from two close modes are compared in Fig. 8 using MAC criterion.

<table>
<thead>
<tr>
<th>No. of mode</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>FDD natural frequency (Hz)</td>
<td>2436</td>
<td>2436</td>
</tr>
</tbody>
</table>

### 3.4 SSI model

SSI method was applied to the time signals of shaft. Stabilization diagram was established as shown in Fig. 9.

![Stabilization diagram](image)

Figure 9: Stabilization diagram.

Stabilization diagram shows the double modes about 2430 Hz (Table 3). Comparison of the mode shapes are shown in Fig. 10.

<table>
<thead>
<tr>
<th>No. of mode</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>SSI natural frequency (Hz)</td>
<td>2430</td>
<td>2442</td>
</tr>
</tbody>
</table>

![MAC Matrix](image)

Figure 10: MAC Matrix between First SSI Two Mode Shapes.
The mode shapes are correlated in some degree. As it is shown in Thomson (1981), if two independent mode shapes exist for double modes, any linear combination of them is also a mode shape. Therefore in practice the two mode shapes may be correlated in some degree.

4 CONCLUSIONS

In this paper the double modes of a gearbox shaft are detected using OMA methods. The FEM model of shaft shows the double modes which can not be identified by conventional modal testing. It is shown that two methods of OMA namely SSI method in time domain and FDD method in frequency domain can detect the double modes. This is due to the inherent nature of SVD to separate the close modes. Also as the structure can be excited at different points and different directions, the double modes can be excited and detected and the mode shapes of the double modes may be correlated in some degree.

REFERENCES


Thomson, W. 1981. Theory of vibration with applications. GEORGE ALLEN & UNWIN LTD.