A Generalized Multiple Random Decrement Method for Modal Parameter Identification of Structures

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ABSTRACT: For the operational modal analysis of civil structures, it is most popular to apply the random decrement method such that the output signal can be transferred into a corresponding free vibration history for the identification of modal parameters. The random decrement method, however, is strictly valid only if the excitation input is a stationary white noise. This assumption is difficult to be satisfied in the case of civil structures and may considerably limit the identification accuracy. In a recent study by the authors, a multiple random decrement method was developed to completely exclude the excitation effect. With the combination of a mode separation technique, this method was successfully applied in the stay cable system where all the modal frequencies are well separated. It is further explored in this study to extend this method for its application to more involved cases where the modal frequencies can be very close.

1 INTRODUCTIONS

The conventional system identification techniques generally need the measurements for both the input excitation and the output structural response to estimate the parameters associated with an assumed structural model. For most of the large-scale civil structures such as high-rise buildings or long bridges, however, it is usually difficult to conduct a forced vibration test where the input force can be accurately manipulated or measured. Considering that the civil structures are naturally subjected to different sources of environmental excitations, the system identification of civil structures can be more conveniently implemented with the ambient vibration measurements. Since only the output signal is available from ambient vibration measurements, the system identification algorithms merely based on output signals are thus required. It is most popular in the literature to first apply the random decrement (RD) method [1-2] such that the corresponding free vibration signal of the system can be extracted from the output signal. Any convenient time-domain method can then be utilized to identify the modal parameters. The RD method, however, is strictly valid only if the excitation input is a stationary white noise. This assumption is difficult to be satisfied in the case of civil structures and may considerably limit the accuracy of identified results.

In a recent study by the authors [3-4], a multiple random decrement (MRD) method was developed to completely exclude the excitation effect in a more general sense. It was suggested to repeatedly apply the RD process on each round of the resulted RD signature to exclusively filter out the substantial effects of the excitation frequencies in the original signal such that the goal of extracting the free vibration time history can be practically attained by relay. With the combination of a mode separation technique, this method was successfully applied in the stay cable system where all the modal frequencies are well separated. It is further explored in this study to extend this MRD method for its application to more involved cases such as building structures where the modal frequencies can be very close to each other.
2 MULTIPLE RANDOM DECREMENT METHOD WITH MODE SEPARATION

The MRD method adopting mode separation techniques is briefly reviewed in this section to serve as the discussion basis of the next section where a generalized MRD method will be proposed.

2.1 Conventional random decrement method

Assume that a linear system with \( n \) degrees-of-freedom (DOF) is subjected to a stationary white noise with a zero mean. The corresponding equations of motion can then be expressed as:

\[
M\mathbf{x}(t) + C\mathbf{x}(t) + K\mathbf{x}(t) = \mathbf{f}(t)
\]

where \( M, C, \) and \( K \) represent the \( n \times n \) structural mass, damping, and stiffness matrix, respectively. In addition, \( \mathbf{x}(t) \) signifies the displacement response vector and \( \mathbf{f}(t) \) is the excitation force vector. If Eq. (1) is satisfied by this system at an instant \( t_i \), a time shifting of \( \tau \) leads to:

\[
M\mathbf{x}(t_i + \tau) + C\mathbf{x}(t_i + \tau) + K\mathbf{x}(t_i + \tau) = \mathbf{f}(t_i + \tau)
\]

Selecting \( N \) different starting instants from Eq. (2) and then computing their mean, it yields:

\[
\begin{align*}
\mathbf{M}\mathbf{y}(\tau) + \mathbf{C}\mathbf{y}(\tau) + \mathbf{K}\mathbf{y}(\tau) &= \frac{1}{N} \sum_{i=1}^{N} \mathbf{f}(t_i + \tau) \rightarrow \mathbf{0} \quad \text{with} \quad \mathbf{y}(\tau) = \frac{1}{N} \sum_{i=1}^{N} \mathbf{x}(t_i + \tau)
\end{align*}
\]

where \( \mathbf{0} \) stands for a zero matrix. Since \( \mathbf{f}(t) \) is assumed to be a zero-mean and stationary white noise, the right-hand side of Eq. (3) must become a zero vector when \( N \rightarrow \infty \). Consequently, Eq. (1) to describe the forced vibration is now turned into another free vibration equation in the form of Eq. (3) where the variable vector \( \mathbf{x}(t) \) is replaced by \( \mathbf{y}(\tau) \). This characteristic function \( \mathbf{y}(\tau) \) is usually called the random decrement signature in the literature.

Taking a signal \( \mathbf{x}(t) \) for example, its RD signature can be obtained with the following steps:

(i) Choose a fixed value \( x_s \) as the cutting threshold such that \( x(t) \) is with values of \( x_s \) at \( N \) different time instants \( t_1, t_2, \ldots, t_N \).

(ii) Set the extracted signal duration \( T_d \) such that the extracted signal can adequately reflect the dynamic characteristics of system.

(iii) Extract \( N \) different time histories, all with a duration \( T_d \), from the measured signal.

Average all those time histories to yield the corresponding RD signature \( \mathbf{y}(\tau) \). Because \( x(t_i) = x_s \) for all \( N \) different time instants, \( y(0) = x_s \) is guaranteed.

Since the initial slopes of the two consecutive extracted time histories are similar in magnitude but opposite in signs, their contributions to the RD signature would be basically cancelled out. Thus, \( y(\tau) \) is a free vibration response induced only by the initial condition \( y(0) = x_s \).

2.2 Multiple random decrement method and mode separation

The RD method can provide effective results for ambient vibration measurements if the frequency content of excitations is not far away from that of a white noise. Unfortunately, this condition is not always satisfied. There usually exist two major problems for the conventional RD method when it is applied to identify the modal parameters of civil structures. First, the filtration of excitation effect may not be as complete as desired if the natural excitation significantly violates the ideal white noise assumption. This fact would lead to a great error in the subsequent identification algorithm where a free vibration history is presumed. On the other hand, the reduced contributions from minor modes during the RD process make it even more difficult to identify the parameters of multiple modes. To deal with these problems, two major modifications from the conventional RD method were recently proposed [3-4] to establish an effective MRD methodology.

It was found [3-4] that the effect of conducting the RD method is to further enhance the peak contribution at the system frequency in the original signal. The resulted RD signature would be almost dominated by the system frequency and corresponds to a free vibration signal if the excitation is close to a white noise. Even for the cases where the excitation force focuses on
certain frequencies, the RD process still guarantees a much higher weighting at the system frequency than the major excitation frequencies. In this situation, the obtained RD signature may not totally exclude the effects from the major excitation frequencies, but would at least significantly deduce their influences. Namely, a perfect free vibration signature may not be immediately obtained in this case after the RD process, but it would at least get close to the desired objective. Based on this concept, a key ingredient for improving the conventional RD method is to repeatedly apply the RD process on each round of the resulted RD signature such that the goal of extracting the free vibration time history can be practically attained by relay.

During the MRD process, the extracted duration $T_d$ in each round of RD process is the length of time history for the next round and thus needs to be taken as large as possible for a sufficient superposition number $N$ in the next round. But contradictorily, no adequate number $N$ would be attained if a very large value of $T_d$ is selected in the current round. To balance this dilemma, it was suggested [3-4] to choose $T_d$ as a half length of the signal from the previous round. Furthermore, the undesired side-effect of the MRD method in suppressing the contributions from minor modes makes it impossible to directly obtain meaningful parameters for several modes at one time. For the cases like stay cables where all the modal frequencies are well separated and the mutual interactions can be negligible, this obstacle was eliminated with the application of mode separation techniques [3-4]. In other complicated cases such as building structures where the modal frequencies can be very close to each other and may have significant mutual interactions, however, the above mode separation technique is not effective anymore. Accordingly, the MRD method is generalized in this study for more extensive applications and will be described in the next section.

3 GENERALIZED MULTIPLE RANDOM DECREMENT METHOD

3.1 Estimation of mode shape vectors

Consider a structure with $n$ DOF's and assume that the displacement measurements $y_1(t)$, $y_2(t)$, ..., $y_n(t)$ for all its DOF's can be obtained in the time domain. These signals include the contributions from all the vibration modes and are eligible to be expressed as

$$
y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_m \\ \phi_1 & \phi_2 & \cdots & \phi_m \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1 & \phi_2 & \cdots & \phi_m \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_m(t) \end{bmatrix} = \Phi \mathbf{z}(t) \quad (4)
$$

where $z_1(t)$, $z_2(t)$, ..., $z_m(t)$ are the time histories of modal displacements and $\phi_1$, $\phi_2$, ..., $\phi_m$ represent the corresponding mode shape vectors. In addition, $y(t)$ and $\mathbf{z}(t)$ signify the column vectors formed by $y_1(t)$, $y_2(t)$, ..., $y_n(t)$ and $z_1(t)$, $z_2(t)$, ..., $z_m(t)$, respectively. Moreover, $\Phi$ denotes the matrix including $\phi_1$, $\phi_2$, ..., $\phi_m$ as column vectors and $\phi_i$ is the component associated with the $i$-th DOF for the $k$-th mode vector. If Fourier transform is further taken on Eq. (4), the displacement vector in the time domain can then be transferred into the frequency domain as

$$\mathbf{Y}(\omega) = \begin{bmatrix} Y_1(\omega) \\ Y_2(\omega) \\ \vdots \\ Y_n(\omega) \end{bmatrix} = \begin{bmatrix} \Phi & \Phi & \cdots & \Phi \end{bmatrix} \begin{bmatrix} Z_1(\omega) \\ Z_2(\omega) \\ \vdots \\ Z_m(\omega) \end{bmatrix} \quad (5)
$$

where $Y_1(\omega)$, $Y_2(\omega)$, ..., $Y_n(\omega)$ represent the Fourier transforms of $y_1(t)$, $y_2(t)$, ..., $y_n(t)$ and $Z_1(\omega)$, $Z_2(\omega)$, ..., $Z_m(\omega)$ denote the Fourier transforms of $z_1(t)$, $z_2(t)$, ..., $z_m(t)$.

Since the Fourier transform $Z_k(\omega)$ of the $k$-th modal displacement in Eq. (5) is simply related to the natural frequency $\omega_k$ and damping ratio $\xi_k$ of the $k$-th mode, its peak amplitude would typically occur at the modal frequency $\omega_k$ in others words, it reaches its maximum contribution at or near $\omega = \omega_k$ and is usually trivial at the other frequencies. More specifically, the contribution to a response signal from each mode may be mixed up at any instant in the time
domain, but can be clearly separated to locate at different modal frequencies in the frequency domain. This feature is essentially valid as long as the excitation does not concentrate at few specific frequencies and the mutual interactions between any two different modes are not considerable.

Based on Eq. (5) and the above discussion, the frequency response can be easily decomposed at different modal frequencies which are well separated. That is, when \( \omega \to \omega_k \),

\[
\mathbf{Y} (\omega_k) = \begin{bmatrix} Y_1(\omega_k) \\ Y_2(\omega_k) \\ \vdots \\ Y_n(\omega_k) \end{bmatrix} \equiv \Phi_k \mathbf{Z}_k (\omega_k) = \begin{bmatrix} \phi_{1k} \\ \phi_{2k} \\ \vdots \\ \phi_{nk} \end{bmatrix} \mathbf{Z}(\omega_k), \quad k = 1, 2, \cdots, n
\]

(6)

With Eq. (6), it leads to

\[
\dot{\Phi}_k = \begin{bmatrix} \phi_{1k} / \phi_{jk} \\ \phi_{2k} / \phi_{jk} \\ \vdots \\ \phi_{nk} / \phi_{jk} \end{bmatrix} \equiv \begin{bmatrix} Y_1(\omega_k) / Y_j(\omega_k) \\ Y_2(\omega_k) / Y_j(\omega_k) \\ \vdots \\ Y_n(\omega_k) / Y_j(\omega_k) \end{bmatrix}, \quad k = 1, 2, \cdots, n
\]

(7)

Eq. (7) evidently indicates that the shape vector for each mode can be directly estimated by dividing the Fourier transforms of simultaneous measurements for different DOF’s at each modal frequency by the Fourier transform of a specific measurement. These ratios should be close to real numbers and provide a good approximation for mode shape vectors when the desired modes are reasonably separated. It should also be noted that the common dividing factor \( Y_j(\omega_k) \) can be chosen to be any particular complex-valued quantity among \( Y_1(\omega_k), Y_2(\omega_k), \ldots, Y_j(\omega_k) \).

### 3.2 Mode convergence control based on initial conditions in random decrement procedures

For the cases where the modal frequencies may be close and have significant mutual interactions, the estimation of mode shape vectors described in the previous subsection may not produce accurate results, but still provide an acceptable initial guess. The next key issue to deal with these more involved cases is how to design an effective algorithm such that the RD signatures corresponding to all the measurements can gradually converge to those with the contribution merely from a single mode during the MRD process. The effects of initial conditions on the response time histories of an MDOF system under free vibration is first examined in this subsection to develop this mode convergence algorithm. The discussion herein is only focused on the influence of initial displacements because the effect of initial velocities is similar.

According to Eq. (4), the initial displacement vector can be decomposed into:

\[
y (0) = \Phi \mathbf{z} (0) = \phi_{1} z_1 (0) + \phi_{2} z_2 (0) + \cdots + \phi_{n} z_n (0)
\]

(8)

where the matrix of mode shape vectors \( \Phi \) is assumed to be normalized with respect to the mass matrix \( \mathbf{M} \), i.e., the orthogonality condition \( \Phi^T \mathbf{M} \Phi = \mathbf{I} \) holds. Consequently, the initial displacement vector can be further transferred to the modal space as

\[
\Phi^T \mathbf{M} y (0) = \Phi^T \mathbf{M} \Phi \mathbf{z} (0) = \mathbf{z} (0) \quad \text{or} \quad \Phi^T \mathbf{M} y (0) = z_k (0), \quad k = 1, 2, \cdots, n
\]

(9)

Combination of Eqs. (8) and (9) leads to:

\[
z_1 (0) = 0, \quad \cdots, \quad z_{k-1} (0) = 0, \quad z_k (0) = c_k, \quad z_{k+1} (0) = 0, \quad \cdots, \quad z_n (0) = 0
\]

(10)

if the initial displacement approaches to the mode shape of the \( k \)-th mode, i.e., \( y (0) \to c_k \). In
other words, only the initial displacement of the \( k \)-th mode is with a value \( c_k \), while the other initial modal displacements are all zero. Therefore, the free vibration response histories of this system are simply governed by the \( k \)-th mode under such circumstances.

Based on the above analysis, the RD signatures would eventually converge to the contribution from the \( k \)-th mode if the initial conditions in each round of RD during the MRD process resemble the \( k \)-th mode shape vector. With this understanding, it is clear that the most critical component to solve this problem is how to control the RD process such that the initial conditions can approach the previously estimated shape vector of the \( k \)-th mode. Before exploring this possibility, it has to be clarified that the cutting thresholds and extracting procedures are not independently conducted for each individual signal when all the measurements of a system with \( n \) DOF's are simultaneously considered in the RD process. Instead, a particular signal needs to be first selected to set the cutting threshold for extraction and superposition. The RD process for the other \( n-1 \) signals then has to be conducted following exactly the same time instants decided by the RD process for that particular signal to maintain the correlation among all the signals after the RD process. Consequently, we are able to control merely a single initial condition corresponding to a specific DOF during the RD process, while the initial conditions for the other \( n-1 \) DOF's result from the simultaneous RD process and can not be directly manipulated.

A new algorithm is proposed in this study to tackle the difficulty mentioned in the previous paragraph. Assume that the measurement for the \( i \)-th DOF is chosen as the reference signal. This selection would naturally lead to one set of RD signatures from the standard RD process:

\[
\mathbf{r}_i(t) = \left\{ r_{i_1}(t), r_{i_2}(t), \ldots, r_{i_n}(t) \right\}^T, \quad i = 1, 2, \ldots, n
\]

where \( r_{i_0}(0) \) is the assigned cutting threshold for the RD procedures conducted on the measurement of the \( i \)-th DOF and the initial conditions \( r_{i_1}(0), \ldots, r_{i+i_1}(0), r_{i+i_2}(0), \ldots, r_{i+i_n}(0) \) at the other DOF's are determined following the RD process. For convenience, this set of RD signatures can be further normalized as

\[
\mathbf{R}_i(t) = \left\{ R_{i_1}(t), R_{i_2}(t), \ldots, R_{i_n}(t) \right\}^T = \frac{\mathbf{r}_i(t)}{|\mathbf{r}_i(0)|}
\]

where \( |\mathbf{r}_i(0)| \) represents the length of the initial displacement vector \( r_i(0) \). With this normalization, the initial condition \( \mathbf{R}_i(0) \) of \( \mathbf{R}_i(t) \) must be a vector with a unit length. Since any one of the \( n \) different DOF's or measurements can be selected as the reference signal, there are totally \( n \) independent sets of RD signatures to be obtained:

\[
\mathbf{R}(t) = \left[ \mathbf{R}_1(t), \mathbf{R}_2(t), \ldots, \mathbf{R}_n(t) \right]
\]

If the normalized shape vector of the \( k \)-th mode has already been estimated from Eq. (7) to be \( \hat{\mathbf{f}} = \left\{ \hat{f}_{k_1}, \hat{f}_{k_2}, \ldots, \hat{f}_{k_n} \right\} \), an initial displacement vector equal to \( \hat{\mathbf{f}} \) can definitely be resulted from the linear combination of the \( n \) sets of independent vectors \( \mathbf{R}_1(0), \mathbf{R}_2(0), \ldots, \mathbf{R}_n(0) \).

More specifically, the coefficients \( a_1, a_2, \ldots, a_n \) for this linear combination can be easily solved with the following system of linear equations:

\[
\begin{bmatrix}
R_{i_1}(0) & R_{i_2}(0) & \cdots & R_{i_n}(0) \\
R_{i_2}(0) & R_{i_2}(0) & \cdots & R_{i_n}(0) \\
\vdots & \vdots & \ddots & \vdots \\
R_{i_n}(0) & R_{i_2}(0) & \cdots & R_{i_n}(0)
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
\vdots \\
a_n
\end{bmatrix} =
\begin{bmatrix}
\hat{f}_{k_1} \\
\hat{f}_{k_2} \\
\vdots \\
\hat{f}_{k_n}
\end{bmatrix}
\]

Based on these determined coefficients, the linear combination of

\[
S_i(t) = a_1 \mathbf{R}_1(t) + a_2 \mathbf{R}_2(t) + \cdots + a_n \mathbf{R}_n(t)
\]
would generate a set of RD signatures which satisfy the initial condition \( \mathbf{S}_i(0) = \hat{\mathbf{g}} \) and attain the goal to approach the estimated shape vector of the \( k \)-th mode.

### 3.3 Multiple random decrement method convergent to a specific mode

Combining the techniques described in the previous two subsections, a generalized algorithm to perform the MRD process with multiple measurements for an MDOF system is proposed in this study. This new algorithm is equipped with a novel feature that can converge to any specific mode. Assuming that \( n \) independent measurements \( \mathbf{y}(t) = [y_1(t) \ y_2(t) \ \ldots \ y_n(t)]^{T} \) are taken, the detailed steps of this MRD process can be listed as follows:

1. Estimate the normalized shape vector of the first mode based on Eq. (7).
2. Choose the measurement of the first DOF as the reference signal and set a fixed value \( y_1(t)_0 \) as the cutting threshold such that \( y_k(t) \) is with values of \( y_1(t)_0 \) at \( N \) different time instants \( t_1, t_2, \ldots, t_N \). The RD process for the other \( n-1 \) measurements are then conducted following exactly the same time instants \( t_1, t_2, \ldots, t_N \).
3. The duration \( T_d \) for each extracted signal is chosen as one half of the original signal.
4. Extract \( N \) different time histories at the time instants \( t_1, t_2, \ldots, t_N \) for each component of \( \mathbf{y}(t) \), all with a duration \( T_d \). Average the extracted signals for each component to yield one set of RD signatures \( \mathbf{r}_i(t) \).
5. For \( i = 2, 3, \ldots, n \), repeat steps (2) to (4) on the measurement of the \( i \)-th DOF to generate the other \( n-1 \) sets of RD signatures.
6. Apply Eq. (12) and normalize \( \mathbf{r}_1(t), \mathbf{r}_2(t), \ldots, \mathbf{r}_n(t) \) into the \( n \) sets of RD signatures \( \mathbf{R}_1(t), \mathbf{R}_2(t), \ldots, \mathbf{R}_n(t) \) shown in Eq. (13).
7. Solve the linear combination coefficients \( a_1, a_2, \ldots, a_n \) according to Eq. (14).
8. Substitute the determined values of \( a_1, a_2, \ldots, a_n \) into Eq. (15) to obtain the set of RD signatures primarily controlled by the first mode.
9. For \( k = 2, 3, \ldots, n \), repeat steps (2) to (8) to produce the other \( n-1 \) sets of RD signatures respectively controlled by the \( k \)-th mode.

For systems under excitations not close to a white noise, it is usually difficult to obtain truthful free vibration signals. Based on the MRD method described in the previous section, it is then suggested to repeatedly apply the RD process with steps (1) to (9) on each round of the resulted RD signatures until the free vibration histories contributing from each mode can be attained to accurately identify the corresponding modal parameters.

### 4 Numerical Examples

A numerical example is adopted in this section to demonstrate the capability of accurately converging to any specific mode with the generalized MRD method developed in the previous section. A 3DOF shear building with the modal frequencies of 4.3, 4.5 and 4.7Hz is considered. All the 3 modal damping ratios are assigned to be 1%. In addition, an initial displacement vector \([-0.2555 \ 0.2710 \ -0.9828]^{T}\), which uniformly includes the contributions from all the 3 modes, is assumed to induce the free vibration responses of structure. Since all the 3 modal frequencies are very close to each other in this system, the mutual interactions among different modes can be obviously observed form the Fourier amplitude spectra for its displacement time histories at various DOFs, as shown in Fig. 1.

Due to the severe interference among all the 3 modes in this case, the peak frequencies illustrated at the Fourier amplitude spectra for different displacement responses may not be consistent. Under such circumstances, it needs to be noted that the dividing ratio of the Fourier transforms, as shown in Eq. (7), still has to be taken at the same frequency point to estimate the mode shape vector. Also because of the mutual mode interactions, there exit substantial errors for the initially estimated mode shape vectors, as compared to the corresponding analytical solutions in Table 1. Theoretically, the mode shape vectors should be real, but the imaginary parts in these estimated vectors are not negligible. This situation indicates the necessity of apply the convergent MRD algorithm to determine more effective mode shape vectors. In this research, the amplitude of the estimated complex-valued vectors are taken as the control initial
conditions to conduct the subsequent RD process such that a better convergence can be effectively obtained.

Three rounds of RD process are performed for this demonstrative example and the convergent mode shape vectors for each round are listed in Tables 2 to 4. In addition, the modal frequencies and damping ratios identified from the third round of RD signatures with the Ibrahim time domain method are also shown in Table 5. Comparing these results, it is evident that essentially real mode shape vectors convergent to the analytical solutions can be attained only after 2 or 3 rounds of RD process. Moreover, the identified modal frequencies and damping ratios listed in Table 5 are also very close to the theoretical values. Therefore, it is clearly verified that the mode convergent MRD algorithm proposed in this study can be applied to the structural systems with close modal frequencies and still produces results with excellent accuracy.

5 CONCLUSIONS

Estimation of mode shape vectors from the frequency responses at peak frequencies is first proposed in this study. A novel concept of controlling the mode convergence based on the initial conditions in the RD process is then developed. Combining these two techniques, the MRD method can be generalized to analyze the ambient signals of an MDOF structure for effectively identifying the modal frequencies, damping ratios and shape vectors. Using a numerical example of 3DOF system with very close modal frequencies, it is demonstrated in this paper that this generalized MRD method can accurately and efficiently determine the required parameters for any target mode even all the modal responses severely interfere with each other.

REFERENCES


Figure 1: Free vibration displacement responses of 3DOF system.
### Table 1: Initially estimated mode shape vectors of 3DOF system from free vibration histories

<table>
<thead>
<tr>
<th>Mode</th>
<th>Fourier Transform with Peak Amplitude Value</th>
<th>Modal Vector</th>
<th>Ratio from FT</th>
<th>Analytical Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1&lt;sup&gt;st&lt;/sup&gt;</td>
<td>at 4.3002Hz</td>
<td>31.51+1.36i</td>
<td>-91.52+30.06i</td>
<td>-162.52-32.88i</td>
</tr>
<tr>
<td>2&lt;sup&gt;nd&lt;/sup&gt;</td>
<td>at 4.5002Hz</td>
<td>-0.3987+0.074i</td>
<td>-0.4752-0.345i</td>
<td>-0.5521</td>
</tr>
<tr>
<td>3&lt;sup&gt;rd&lt;/sup&gt;</td>
<td>at 4.7002Hz</td>
<td>-118.27-8.70i</td>
<td>107.80-8.63i</td>
<td>-63.60+49.58i</td>
</tr>
</tbody>
</table>

### Table 2: Convergent 1<sup>st</sup> mode shape vector of 3DOF system for each round of RD

<table>
<thead>
<tr>
<th>MRD</th>
<th>Fourier Transform with Peak Amplitude Value</th>
<th>Modal Vector</th>
<th>Ratio from FT</th>
<th>Analytical Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1&lt;sup&gt;st&lt;/sup&gt; Round</td>
<td>at 4.3004Hz</td>
<td>65.35+13.31i</td>
<td>187.64+37.19i</td>
<td>307.66+55.77i</td>
</tr>
<tr>
<td>2&lt;sup&gt;nd&lt;/sup&gt; Round</td>
<td>at 4.3208Hz</td>
<td>54.15+26.66i</td>
<td>158.42+78.62i</td>
<td>249.99+122.42i</td>
</tr>
<tr>
<td>3&lt;sup&gt;rd&lt;/sup&gt; Round</td>
<td>at 4.3217Hz</td>
<td>55.21+24.00i</td>
<td>162.49+70.91i</td>
<td>256.75+112.63i</td>
</tr>
</tbody>
</table>

### Table 3: Convergent 2<sup>nd</sup> mode shape vector of 3DOF system for each round of RD

<table>
<thead>
<tr>
<th>MRD</th>
<th>Fourier Transform with Peak Amplitude Value</th>
<th>Modal Vector</th>
<th>Ratio from FT</th>
<th>Analytical Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1&lt;sup&gt;st&lt;/sup&gt; Round</td>
<td>at 4.5004Hz</td>
<td>-136.55+7.02i</td>
<td>-150.31+17.21i</td>
<td>281.84+27.25i</td>
</tr>
<tr>
<td>2&lt;sup&gt;nd&lt;/sup&gt; Round</td>
<td>at 4.5209Hz</td>
<td>-105.28+67.88i</td>
<td>-122.66+81.05i</td>
<td>224.99+122.42i</td>
</tr>
<tr>
<td>3&lt;sup&gt;rd&lt;/sup&gt; Round</td>
<td>at 4.4817Hz</td>
<td>-114.25+46.48i</td>
<td>-134.31+54.48i</td>
<td>242.75+98.52i</td>
</tr>
</tbody>
</table>

### Table 4: Convergent 3<sup>rd</sup> mode shape vector of 3DOF system for each round of RD

<table>
<thead>
<tr>
<th>MRD</th>
<th>Fourier Transform with Peak Amplitude Value</th>
<th>Modal Vector</th>
<th>Ratio from FT</th>
<th>Analytical Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1&lt;sup&gt;st&lt;/sup&gt; Round</td>
<td>at 4.7004Hz</td>
<td>235.72+39.48i</td>
<td>90.03+31.26i</td>
<td>1</td>
</tr>
<tr>
<td>2&lt;sup&gt;nd&lt;/sup&gt; Round</td>
<td>at 4.6809Hz</td>
<td>212.85+75.46i</td>
<td>93.97+36.68i</td>
<td>1</td>
</tr>
<tr>
<td>3&lt;sup&gt;rd&lt;/sup&gt; Round</td>
<td>at 4.7218Hz</td>
<td>230.10+30.38i</td>
<td>-197.60+26.51i</td>
<td>102.13+13.88i</td>
</tr>
</tbody>
</table>

### Table 5: Identified modal parameters of 3DOF system after 3 rounds of RD

<table>
<thead>
<tr>
<th>MRD</th>
<th>1&lt;sup&gt;st&lt;/sup&gt; Mode</th>
<th>2&lt;sup&gt;nd&lt;/sup&gt; Mode</th>
<th>3&lt;sup&gt;rd&lt;/sup&gt; Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>1&lt;sup&gt;st&lt;/sup&gt; Round</td>
<td>Frequency (Hz)</td>
<td>4.301</td>
<td>4.300</td>
</tr>
<tr>
<td>2&lt;sup&gt;nd&lt;/sup&gt; Round</td>
<td>Damping Ratio (%)</td>
<td>1.001</td>
<td>0.992</td>
</tr>
</tbody>
</table>