FEM SENSITIVITY BASED OPERATIONAL SHAPE EXPANSION AND SYSTEM IDENTIFICATION

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ABSTRACT

A recently introduced FEM sensitivity vector based operational shape expansion technique forms the basis of strategies for system identification opportunities. In the case of a linear system, operational modes (shapes, natural frequencies and damping) may be identified by random decrement methodology or cross-spectral analysis. When moderate, localized nonlinearities are present, operational modes may be successfully estimated (as demonstrated in past ISS component modal testing). The identified mode shapes, at measurement locations are then expanded to the FEM grid set using FEM sensitivity vectors. Identification of reconciled FEM parameters is then accomplished using a Monte-Carlo strategy that minimizes a sensitivity-based cost function. Two alternative cost functions, namely (1) a form requiring a reduced order (TAM) mass matrix and (2) a new form the does not employ a TAM mass matrix appear to effectively estimate unknown test article parameters. The new cost function form, while slightly less accurate than the older form, provides for a wider range of system identification opportunities in situations where a TAM mass matrix is not preferred or not easily calculated.

Keywords: FEM, Sensitivity, Shape, Expansion, System, Identification

1. INTRODUCTION

An accurate, efficient method for computation of structural dynamic modal frequency and mode shape sensitivities due to variation of stiffness and mass parameters was published at IMAC XXIX [1]. At the heart of the method is the formation of residual vectors that describe the distributed effect of both stiffness and mass deviations with respect to the baseline system’s modes. The combined set of baseline modes and residual vectors (residual enhanced sensitivity vectors or RESVs) define a transformation matrix which is used to form a greatly reduced order sensitivity model that accurately tracks changes in system modes and natural frequencies driven by large stiffness and mass variations. This technique was successfully applied to effect reconciliation of an International Space Station (ISS) component’s finite element model with multi-shaker induced modal test data, utilizing Monte-Carlo minimization of a balanced cost function, as early as 2001[2]. A noteworthy aspect of the ISS

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experience is that the presence of nonlinearity in the measured system does not necessarily compromise validity of the methodology.

Variations of the above cited sensitivity and system identification strategy, employing response based or hybrid modal/response cost functions[3], may be successfully utilized using operational (ambient) modal test data or base excited modal test data, as long as such data is sufficiently complete and accurate[4]. Challenges associated with operational modal data occur due to sparse allocation of accelerometers [Note that accelerometer allocation in aircraft and spacecraft modal testing (especially in the United States) is substantially more aggressive due to test-analysis model (TAM) mass matrix defined measured mode orthogonality criteria[5].]

The present paper explores modifications to the previously established sensitivity and system identification strategy that circumvent measured mode orthogonality criteria. One feature of the methodology, namely extension of the SEREP procedure, was recently presented at IMAC XXXI[6]. The second new feature, introduced herein, consists of a modified modal cost function that does not require a TAM mass matrix.

2. SENSITIVITY VECTORS AND MEASURED MODE EXPANSION

The FEM matrix equation set describing free vibration of a baseline or altered undamped system is

\[
\begin{bmatrix}
    M_0 + \sum_{i=1}^{N} p_i \cdot \Delta M_i \\
    K_0 + \sum_{i=1}^{N} p_i \cdot \Delta K_i
\end{bmatrix} \{\ddot{u}\} + \begin{bmatrix}
    M_0 + \sum_{i=1}^{N} p_i \cdot \Delta M_i \\
    K_0 + \sum_{i=1}^{N} p_i \cdot \Delta K_i
\end{bmatrix} \{u\} = \{0\}
\]

(1)

\(M_0\) and \(K_0\) are the ‘core’ mass and stiffness matrices, respectively. For designated reference alterations (\(i=1\) to \(N\)), \(\Delta M_i\) and \(\Delta K_i\) are specific mass and stiffness change partitions, respectively, and \(p_i\) are the unknown parameters associated with the mass and stiffness changes. \(\{u\}\) are the FEM physical displacements.

The low frequency undamped modes of the baseline system are solutions of the eigenvalue problem

\[
\begin{bmatrix}
    M_0 + \sum_{i=1}^{N} p_{i0} \cdot \Delta M_i \\
    K_0 + \sum_{i=1}^{N} p_{i0} \cdot \Delta K_i
\end{bmatrix} \{\ddot{u}\} + \begin{bmatrix}
    M_0 + \sum_{i=1}^{N} p_{i0} \cdot \Delta M_i \\
    K_0 + \sum_{i=1}^{N} p_{i0} \cdot \Delta K_i
\end{bmatrix} \{u\} = \{0\}
\]

(2)

where \([\Phi_{0L}]\) and \([\lambda_{0L}]\) are the eigenvectors and eigenvalues, respectively, and \(p_{i0}\) are values of the parameters for the baseline model.

Definition of residual vectors describing parametric variations in Equation 1 is accomplished utilizing the lowest frequency mode shapes of the baseline structure as well as the lowest mode shapes associated with each independent alteration of the structure

\[
\begin{bmatrix}
    K_0 + p_l \Delta K_l \Phi_{dl} \\
    M_0 + p_l \Delta M_l \Phi_{dl} \lambda_{dl}
\end{bmatrix} = \{0\} \quad \text{(for } i=1, \ldots, N),
\]

(3)

where \([\Phi_{dl}]\) and \([\lambda_{dl}]\) are the altered system eigenvectors and eigenvalues, respectively. Note that each \(p_l\) may be a finite (rather than infinitesimal parametric perturbation) with respect to \(p_{i0}\). The initial set of trial vectors that redundantly encompass all low frequency altered system mode eigenvectors is

\[
[p_l \Phi_{dl} \Phi_{2l} \ldots \Phi_{Nl}]
\]

(4)

The approximate generalized sensitivity model (that may be employed in a more complete system identification exercise) is
\[
\begin{bmatrix}
k_0 + \sum_{i=1}^{N} p_i [\Delta k_i]\end{bmatrix}[\phi] - \begin{bmatrix}m_0 + \sum_{i=1}^{N} p_i [\Delta m_i]\end{bmatrix}[\phi]\lambda] = [0],
\]
(5)
where the respective reduced stiffness and mass matrix components are
\[
[k_0] = [\Phi_{OL}^T K_0 \Phi_{OL}],
[m_0] = [\Phi_{OL}^T M_0 \Phi_{OL}],
[\Delta k_i] = [\Phi_{OL}^T \Delta k_i \Phi_{OL}],
[\Delta m_i] = [\Phi_{OL}^T \Delta M_i \Phi_{OL}]
\]
(6)
and \([\phi]\) and \([\lambda]\) are the reduced system eigenvectors and eigenvalues, respectively.

The low frequency physical modes for the altered dynamic system are recovered using the relationship
\[
[\Phi_L] = [\Phi_{OL}][\phi]
\]
(7)

3. BALANCED COST FUNCTION DEFINITIONS

3.1. Basic modal cost function

Consider the standard expression for the undamped structural dynamics eigenvalue problem,
\[
[K][\Phi] - [M][\Phi][\lambda] = 0
\]
(8)
where \([K]\), \([M]\), \([\Phi]\), and \([\lambda]\) are the FEM mass, stiffness, eigenvectors and eigenvalues, respectively.

When modal test data (designated by subscript ‘T’) is substituted into the above expression,
\[
[K][\Phi_T] - [M][\Phi_T][\lambda_T] = [R]
\]
(9)
there is a residual error, \([R]\), due to (a) differences between the FEM and modal test data and (b) measurement error. Pre-multiplication by the FEM mode shapes results in
\[
[\Phi^T][K][\Phi_T] - [\Phi^T][M][\Phi_T][\lambda_T] = [\Phi^T][R]
\]
(10)
Noting the transpose of \([K][\Phi]\) from equation (8), the modal cost function, \([C_{\Phi\lambda}]\), is defined as
\[
[C_{\Phi\lambda}] = [\lambda]^{-1}[\Phi^T][R] = [C] - [\lambda]^{-1}[C][\lambda_T]
\]
(11)
where
\[
[C] = [\Phi^T][M][\Phi_T]
\]
(12)
is the cross-orthogonality matrix between test and FEM mode shapes. When there are rigid body and/or zero-frequency mechanism modes present, a shift operator, \([\lambda_s]\) is employed to avoid numerical problems. The shifted modal cost function is
\[
[C_{\Phi\lambda}] = [C] - [\lambda + \lambda_s]^{-1}[C][\lambda_T + \lambda_s]
\]
(13)
The balanced cost function, \([C_{\Phi\lambda}]\), is null when test and FEM modes coincide. Therefore, iteration on FEM parameter values to search for the minimum norm of \([C_{\Phi\lambda}]\) identifies the appropriate system model. Experience with actual measured modal data, which had noise and localized non-linearity\(^2\), demonstrated that Monte-Carlo iteration converges to well-defined solutions.

3.2. Implementation with a TAM and measured degrees of freedom

The earliest implementation of the basic cost function for an ISS component modal test\(^2\) utilized a TAM mass matrix, \([M_{test}]\), defined by Guyan reduction\(^7\), and modal vectors, \([\Phi_T]_{test}\), deduced at measured degrees of freedom. The FEM modes, associated with parametric sensitivity (described in Section 2 of this paper) are recovered utilizing the FEM partition corresponding to measured degrees of freedom, as follows:
\[
[\Phi]_{test} = [\Phi_{OL}]_{test}[\phi]
\]
(14)
Therefore, the cross-orthogonality matrix (initially defined by equation 12) takes on the approximate form,

\[
[C] = [\Phi]^T_{\text{test}} \begin{bmatrix} M_{\text{test}} & [\Phi]_{\text{test}} \end{bmatrix}
\]

(15)

3.3. Implementation with SEREP measured degrees of freedom

When a TAM is not available, the generalized modal coefficients associated with measured modes, \([\varphi]_{\text{test}}\), in equation 14 may be estimated using the \(^\text{“SEREP”}\) \(^\text{[8]}\) unweighted least squares calculation

\[
[\varphi]_{\text{test}} = \left[ [\Phi]_{\text{test}}^T \begin{bmatrix} [\Phi]_{\text{test}} \end{bmatrix} \right] ^{-1} \left[ [\Phi]_{\text{test}}^T \begin{bmatrix} [\Phi]_{\text{test}} \end{bmatrix} \right] [\varphi]_{\text{test}}
\]

(16)

The expanded measured mode vectors in FEM degrees of freedom are therefore estimated as

\[
[\Phi]_{\text{test}} = \begin{bmatrix} [\Phi]_{\text{OL}} \end{bmatrix}
\]

(17)

By employing the approximate low frequency FEM modes as defined in equation 7, and the baseline FEM mass matrix, the cross-orthogonality matrix takes the form

\[
[C] = [\Phi]_{\text{test}}^T \begin{bmatrix} M_{\text{test}} & [\Phi]_{\text{test}} \end{bmatrix} = [\varphi]^T \begin{bmatrix} [\Phi]_{\text{OL}}^T M_{\text{OL}} [\Phi]_{\text{OL}} \end{bmatrix} [\varphi] = [\varphi]^T \begin{bmatrix} m_{\varphi} \end{bmatrix}
\]

(18)

4. ILLUSTRATIVE EXAMPLES

4.1. FEM baseline and sensitivity model

A four segment beam, illustrated below in Figure 1, is used as an illustrative example (described by a 73 grid point FEM) to demonstrate the measured mode expansion procedure.

**Figure 1: Illustrative Example Four Segment Beam**

Beam sections 1-3 each have flexural stiffness, \(EI=1.473\times10^8\), and mass per unit length, \(\rho A=0.00122\). Beam section 4, a small appendage, has flexural stiffness, \(EI=7.363\times10^6\), and mass per unit length, \(\rho A=0.0122\). The reference rotational bending stiffness values for the sensitive interfaces are \(\Delta K_1=\Delta K_2=\Delta K_3=10^7\). Note that the first stiffness is intended to represent the beam’s foundation, and the other two rotational stiffnesses represent interior joints.

Modes associated with four different combinations of local rotational bending stiffness parameters are calculated as follows:

1. Baseline model: \(p_1=5, p_2=p_3=10\) (stiff interior joints)
2. Sensitivity model 1: \(p_1=5, p_2=3, p_3=1\) (soft interior joints)
3. Sensitivity model 2: \(p_1=5, p_2=1, p_3=3\) (soft interior joints)
4. Simulated test data: \(p_1=5, p_2=0.50, p_3=0.25\)

The computed modal frequencies associated with the above noted combinations of parameters are shown below in Table 1.
Table 1: Segmented Beam Modal Frequencies

<table>
<thead>
<tr>
<th>Mode</th>
<th>Baseline (Hz)</th>
<th>Sensitivity 1 (Hz)</th>
<th>Sensitivity 2 (Hz)</th>
<th>&quot;Test&quot; (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.339</td>
<td>1.269</td>
<td>1.156</td>
<td>0.967</td>
</tr>
<tr>
<td>2</td>
<td>8.960</td>
<td>6.829</td>
<td>7.137</td>
<td>4.602</td>
</tr>
<tr>
<td>3</td>
<td>23.815</td>
<td>17.709</td>
<td>18.766</td>
<td>11.228</td>
</tr>
<tr>
<td>4</td>
<td>49.584</td>
<td>49.450</td>
<td>49.496</td>
<td>49.367</td>
</tr>
<tr>
<td>5</td>
<td>59.229</td>
<td>57.365</td>
<td>57.636</td>
<td>65.642</td>
</tr>
</tbody>
</table>

It is of interest to view the baseline model and simulated modes shapes illustrated below in Figure 2.

Note that the second and third modes of the simulated test model are characterized by abrupt changes in slope due to the “soft” joints, which contrast the smooth changes in the baseline model, which has stiff joints. The modes associated with the two sensitivity cases (not shown) also are characterized by abrupt changes in slope due to “soft” joint combinations.

In order to simulate measured operating deflection shapes, lateral deflections at the eight grid points denoted in Figure 3 were selected.

Trial vectors for expansion of five simulated operating deflection shapes were generated using the following strategies:

1. 5 baseline system (stiff joints) modes (classic SEREP application)
2. Guyan expansion using the baseline model stiffness (stiff joints)
3. 7 trial vectors consisting of 5 baseline modes and 2 residual vectors (equations 1-7)

Modal expansions resulting from the above three trial vector sets are illustrated in Figure 4 for the second and third measured test modes (operating deflection shapes). It is clear in these results that the trial vectors associated with sensitivity analyses produce features that express the abrupt slope changes that the two other trial vector sets cannot express.
A more precise numerical assessment of the three modal expansion strategies is realized by calculation of (a) orthogonality matrices among expanded modes, and (b) cross-orthogonality matrices between “exact” and expanded modes. The orthogonality and cross-orthogonality relationships are:

\[
[\text{OR}] = [\Phi_T]^T [M_O] [\Phi_T], \quad [\text{COR}] = [\Phi]^T [M_O] [\Phi_T]
\]  

(19)

<table>
<thead>
<tr>
<th>[OR] for Expanded Modal Vectors</th>
<th>[COR] with respect to Exact Modes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sensitivity Vectors (SEREP)</td>
<td>Sensitivity Vectors (SEREP)</td>
</tr>
<tr>
<td>100 0 0 0 0 0</td>
<td>100 0 0 0 0 0</td>
</tr>
<tr>
<td>0 100 0 0 0 0</td>
<td>0 100 0 0 0 0</td>
</tr>
<tr>
<td>0 0 100 0 0 0</td>
<td>0 0 100 0 0 0</td>
</tr>
<tr>
<td>0 0 0 100 0 0</td>
<td>0 0 0 100 0 0</td>
</tr>
<tr>
<td>0 0 0 0 100 0</td>
<td>0 0 0 0 100 0</td>
</tr>
<tr>
<td>0 0 0 0 0 100 0</td>
<td>0 0 0 0 0 100 0</td>
</tr>
</tbody>
</table>

Table 2: Orthogonality and Cross-Orthogonality Matrices for Expanded Operating Deflection Shapes

The modal expansion comparisons with “exact” modal vectors illustrated in Figure 4 and Table 2 clearly indicate that utilization of sensitivity based trial vectors provide superior fidelity for the SEREP expansion process. This is attributed to the fact that the sensitivity based residual vectors possess the (abrupt slope change) features required to reconstruct the behavior of the “softened” joints.

4.2. System identification using balanced cost functions and Monte-Carlo iteration

Performance of system identification (estimation of system parameters) for the segmented beam model is evaluated for three forms of the balanced cost function, namely:

1. Exactly computed FEM modes vs. expanded (simulated) test modes
2. Sensitivity implementation using TAM and (simulated) test modes (weighted)
3. Sensitivity implementation using SEREP and (simulated) test modes (unweighted)

In each of the above three forms, the balanced cost function norm is defined as

\[
\text{ERR} = |C_{01}| = \sum_{i,j} |C_{0\hat{i}}|_{ij}
\]

(20)

The Monte-Carlo iteration search patterns for the each of the two sensitivity parameters are illustrated below.
Values for the FEM sensitivity parameters and estimated modal frequencies for the three balanced cost function forms, along with the exact system parameters are summarized below.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>FEM Calculation (Exact)</th>
<th>Full FEM Sensitivity</th>
<th>Weighted RESV</th>
<th>Unweighted RESV</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Value</td>
<td>Value</td>
<td>Error (%)</td>
<td>Value</td>
</tr>
<tr>
<td>P2</td>
<td>0.5000</td>
<td>0.5000</td>
<td>0.007</td>
<td>0.4992</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.4583</td>
</tr>
<tr>
<td>P3</td>
<td>0.2500</td>
<td>0.2507</td>
<td>0.280</td>
<td>0.2474</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.2469</td>
</tr>
<tr>
<td>Mode</td>
<td>Freq (Hz)</td>
<td>Freq (Hz)</td>
<td>Error (%)</td>
<td>Freq (Hz)</td>
</tr>
<tr>
<td>1</td>
<td>0.987</td>
<td>0.987</td>
<td>0.01</td>
<td>0.986</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.958</td>
</tr>
<tr>
<td>2</td>
<td>4.602</td>
<td>4.606</td>
<td>0.08</td>
<td>4.588</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>4.581</td>
</tr>
<tr>
<td>3</td>
<td>11.228</td>
<td>11.232</td>
<td>0.03</td>
<td>11.211</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>11.038</td>
</tr>
<tr>
<td>4</td>
<td>49.367</td>
<td>49.367</td>
<td>0.00</td>
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<td></td>
<td>49.365</td>
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<tr>
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<td>55.642</td>
<td>55.642</td>
<td>0.00</td>
<td>55.638</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>55.597</td>
</tr>
</tbody>
</table>

The above results indicate that the weighted mode expansion form (TAM, equation 15) yields better accuracy than the unweighted mode expansion form (SEREP, equation 18).

A final objective evaluation of the fidelity of the identified model is realized by substitution of the estimated system parameters into the FEM with subsequent comparison of modes of the FEM models. The cross-orthogonality matrices relating the models based on the three versions of the balanced cost function are virtually identical. Therefore the identified models appear to be equally capable of accurate characterization of forced responses and stresses.

### 4.3. System identification of the ISS P-5 modal test article

The above discussed example of a segmented beam provides an ideal illustration of the performance of system identification strategies with simulated measured data. It is useful, in addition, to review results from the first experimental application of the general sensitivity and system identification methodology on the ISS P-5 modal test article in 2001. The project was planned and executed by Boeing Rocketdyne Division, NASA Marshall Space Flight Center (MSFC), and Measurement Analysis Corporation. Modal testing, conducted at NASA/MSFC, employed multi-shaker random excitation (photograph and measured response data shown below).
The primary difficulty encountered during the model test was associated with nonlinearity in several trunnion joints, as evidenced by harmonic distortion and reduced coherence (illustrated in the graphs to the right of the above photograph). The difficulty was compounded by the fact that successive multi-shaker random tests produced modes which were slightly different from one another in frequency and shape (with respect to previous tests). It was determined by NASA (the sponsoring agency) that reconciliation of the ISS P-5 FEM and modal test would be judged acceptable if agreement could be reached for three separate modal test sets (TSS2, TSS4, TSS17). This was accomplished employing Monte-Carlo system identification (implementation with TAM) as summarized below in Table 4, which provides pre-test, post-test (reconciled) and test natural frequencies, along with the cross-orthogonality coefficient, COR (% units), between post-test and test mode shapes.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Pre Test</th>
<th>Post Test</th>
<th>Test</th>
<th>COR</th>
<th>Post Test</th>
<th>Test</th>
<th>COR</th>
<th>Post Test</th>
<th>Test</th>
<th>COR</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>18.67</td>
<td>16.48</td>
<td>16.94</td>
<td>98</td>
<td>15.31</td>
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<td>17.31</td>
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<td>98</td>
</tr>
<tr>
<td>2</td>
<td>19.31</td>
<td>18.08</td>
<td>17.58</td>
<td>97</td>
<td>18.02</td>
<td>17.92</td>
<td>98</td>
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<td>17.61</td>
<td>98</td>
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<td>28.90</td>
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<td>92</td>
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<td>92</td>
</tr>
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<td>31.63</td>
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<td>77</td>
</tr>
<tr>
<td>7</td>
<td>34.01</td>
<td>33.32</td>
<td>33.66</td>
<td>82</td>
<td>32.93</td>
<td>33.49</td>
<td>72</td>
<td>33.71</td>
<td>33.64</td>
<td>89</td>
</tr>
<tr>
<td>8</td>
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<td>34.64</td>
<td>35.19</td>
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</tr>
<tr>
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<td>36.98</td>
<td>38.38</td>
<td>97</td>
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<td>38.58</td>
<td>85</td>
<td>36.99</td>
<td>38.32</td>
<td>97</td>
</tr>
</tbody>
</table>

Table 4: Summary of ISS P-5 Modal Test Results

The ISS P-5 experience demonstrated the ability of a FEM RESV sensitivity model and system identification using balanced cost functions to reconcile modal test and FEM in a particularly challenging situation.

5. CONCLUSIONS

This paper describes methodology, which has been developed and refined over a twelve year period, for reconciliation of complete FEM system models with measured modal data. The methodology was originally developed to address challenges associated with detailed laboratory modal tests following
strict U.S. Department of Defense and NASA standards and criteria. The present paper introduces newer methodology that is suited for both laboratory and field (operational) modal analysis applications. Specifically, a test-analysis model (TAM) mass matrix is not required.

The following specific conclusions are the result of the present and previous related works:

1. Residual enhanced sensitivity vectors (RESV) form a more robust basis than baseline FEM modes for measured mode shape expansion and model order reduction.
2. Reduced dynamic models (mass and stiffness) based on RESV closely approximate complete FEM parametric changes in mode shape and natural frequency.
3. Monte-Carlo iteration using balanced cost functions effectively identifies test reconciled FEMs.
4. A new form of the balanced cost function is useful for system identification (FEM reconciliation) in situations for which a TAM is not available, practical or desired (e.g., operational modal analysis field applications).
5. The identified, reconciled FEM provides for accurate forced response and stress prediction, principally due to the fact that a complete FEM (rather than reduced dynamic model) is realized.

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REFERENCES

APPENDIX A: PROCEDURE FOR GENERATING RESIDUAL VECTOR SETS
Consider the general task of relating an arbitrary set of trial vectors, [Ψ], to a set of the lowest frequency system modes, [ΦOL], of an undamped dynamic system

\[
\begin{bmatrix}
K_o \\
M_o \\
\rho_{ol}
\end{bmatrix}
[\Phi_{ol}] = [0] \quad \text{where} \quad [\Phi^T_{ol} M_o \Phi_{ol}] = [I] \quad \text{and} \quad [\Phi^T_{ol} K_o \Phi_{ol}] = [\rho_{ol}] \quad (a1)
\]
In addition, the stiffness matrix is related to the low and high frequency modes according to
\[ [K_O] = [M_O \Phi_{OL} \lambda_{OL}] [\Phi_{OL}^T M_O] + [M_O \Phi_{OL} \lambda_{OL}] [\Phi_{OL}^T M_O] \] (a2)

The set of trial vectors is the sum of (a) a linear combination of system modes, \([\Phi_{OL}]\) and (b) residual vectors, \([\Psi_R]\)
\[ [\Psi] = [\Phi_{OL}] [Q] + [\Psi_R] \] (a3)

Employing weighted least squares, and enforcing orthogonality of the residuals with respect to the system modes,
\[ [\Phi_{OL}^T K_O \Psi] = [\Phi_{OL}^T M_O \Psi] - [\Phi_{OL}^T \lambda_{OL} \Psi] \] , resulting in
\[ [Q] = [\Phi_{OL}^T K_O \Phi_{OL}]^{-1} [\Phi_{OL}^T K_O \Psi] \] (a4)

Thus the residual vectors are,
\[ [\Psi_R] = [\Psi] - [\Phi_{OL}] [\Phi_{OL}^T K_O \Phi_{OL}]^{-1} [\Phi_{OL}^T K_O \Psi] = [\Psi] - [\Phi_{OL}] [\lambda_{OL}]^{-1} [\Phi_{OL}^T K_O \Psi] \] (a5)

Due to the assumption in Equation a3, the residual vectors are orthogonal with respect to the stiffness matrix as well as the mass matrix, as shown below:
\[ [\Phi_{OL}^T M_O \Psi_R] = [\Phi_{OL}^T M_O \Psi] - [\lambda_{OL}]^{-1} [\Phi_{OL}^T M_O \Psi] = [\Phi_{OL}^T M_O \Psi] - [\lambda_{OL}]^{-1} [\Phi_{OL}^T K_O \Psi] \] (a7)

Substituting Equation a2 into the above results in
\[ [\Phi_{OL}^T M_O \Psi_R] = [\Phi_{OL}^T M_O \Psi] - [\lambda_{OL}]^{-1} [\Phi_{OL}^T M_O \Psi] \equiv [0] \] (a8)

It can also be proven that mathematically equivalent residual vectors may be derived using the mass matrix as a weighting matrix,
\[ [\Psi_R] = [\Psi] - [\Phi_{OL}] [\Phi_{OL}^T M_O \Phi_{OL}]^{-1} [\Phi_{OL}^T M_O \Psi] = [\Psi] - [\Phi_{OL}] [\Phi_{OL}^T M_O \Psi] \] (a9)

While the residual vectors are orthogonal to the low frequency modes, they are not necessarily linearly independent of one another. A reduced order linearly independent residual vector set, however, may be estimated using singular value decomposition. This is accomplished by solving the following algebraic eigenvalue problem,
\[ [A] = [\Psi_R^T M_O \Psi_R] , \quad [A] [\lambda_{\psi}] = [\lambda_{\psi}] [\psi] , \quad \lambda_{\psi 1} \geq \lambda_{\psi 2} \geq \lambda_{\psi 3} \geq .... \] (a10)

A suitable cut-off criterion, noted below, that has been employed over the past twelve years with good success in defining the suitable reduced trial vector set, is
\[ \frac{\lambda_{\psi N}}{\lambda_{\psi 1}} \leq 10^{-5} \] (a11)

The set of linearly independent residual vectors and an augmented trial vector set, respectively, are defined as
\[ [\Psi'_R] = [\Psi_R] [\phi_p] \quad \text{and} \quad [\Psi_{OL}'] = [\Phi_{OL} \quad \Psi_R'] \] (a12)

Finally, a mutually orthogonal trial vector set may be defined based on the complete (not truncated) solution of the following reduced eigenvalue problem:
\[ [\Psi_{OL}^T K_O \Psi_{OL}] [\lambda_{\psi}] - [\Psi_{OL}^T M_O \Psi_{OL}] [\lambda_{\psi}] = [0] \] (a13)

where
\[ [\Psi_{OL}] = [\Psi_{OL}'] \] (a14)