IMPLICATIONS OF CLOSELY SPACED MODES IN OMA

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ABSTRACT

It has been known for a long time that closely spaced modes represent a challenge in all identification problems, thus also in operational modal analysis (OMA) the issue is important. In this paper some recent results about the case of closely spaced modes are presented and it is discussed how this knowledge can be used in test-model correlation. It is illustrated that the sensitivity of mode shapes with respect to perturbations of the system gets high when modes get closely spaced, and that the increased sensitivity is mainly due to rotations in the subspace spanned by the mode shapes of the closely spaced modes. An analytical solution for the rotation angle is provided in case of moderately spaced modes where the rotation angle turn out to be proportional to the ratio $\frac{\omega_1}{\Delta \omega}$ between the mean natural frequency $\omega$ of the two considered modes, and the spacing $\Delta \omega$ between them. This solution is valid only for moderate values of the ratio $\frac{\omega_1}{\Delta \omega}$, and therefore, in case of very closely spaced modes, a separate analytical solution is formulated as an eigenvalue problem providing a solution that becomes independent of the ratio $\frac{\omega_1}{\Delta \omega}$. This solution defines an upper bound for the rotation of the mode shapes. The fact that the mode shapes of closely spaced modes mainly rotate in their subspace, indicates that in this case, only the subspace spanned by the mode shapes is important. This means that for closely spaced modes correlation between different identification estimates or between model and experiment should be calculated between subspaces and not between the individual mode shape vectors. This naturally introduces the concept of subspace angles that is presented as a natural way of expressing the correlation between subspaces of closely spaced mode shapes.

Keywords: closely spaced modes, mode shape sensitivity, subspace, mode shape rotation, rotation angle.

1. INTRODUCTION

Normally we consider a mode consisting of an eigenvalue and its corresponding eigenvector as something that is characteristic for a structure.
However, when the difference between two eigenvalues is very small - we say the two modes are closely spaced – it becomes more difficult to treat the two modes as individual quantities, and more natural - and also easier - to consider them together.

In this paper we will investigate the problem of closely spaced modes and how we can treat the two modes as individual quantities and how we can treat them together.

Throughout the literature, there are many remarks on closely spaced modes, but little analysis focused on this problem. Among the many remarks or statements on closely spaced modes, the following can be considered to be more or less widely accepted:

- In modal testing it is known that multiple input is needed to see closely spaced modes, and in order to make the identification more robust one might like to perturb the test specimen
- For structural models it is known that closely spaced modes are highly sensitive to small perturbations of the mass and stiffness contributions
- In model validation and updating it is known that mode shape pairing is an important and difficult issue, and therefore other methods like subspace angles might be considered

The first statement is not the focus of this paper. Multiple input is essential in all modal testing, but in OMA this comes for free when using the full correlation function matrix and making sure that input is reasonably random in time and space Brincker and Ventura [1]. The other issue of the first statement is a well-known way of treating the difficult testing problem when having a set of closely spaced modes where the mode shapes are over sensitive to small changes in the test setup. The problem can be overcome perturbing the system by adding some masses in order to enlarge the frequency spacing \( \Delta \omega \) between the two closely spaced modes, Motterhead et al. [2,3].

In the rest of this paper we shall focus on the two last of the above mentioned statements. First we will give a theoretical background for the increased sensitivity of mode shapes, then we will briefly discuss what it means that two modes are closely spaced, and finally we will consider a few ways to perform test-model correlation in cases of closely spaced modes.

## 2. CASE OF REPEATED POLES

The case of closely spaced eigenvalues has inherited important basic properties from the case of repeated eigenvalues, and therefore it is useful to revisit this well-known case as an introduction to the subject. Considering the theory of repeated eigenvalues we will limit the analysis to the simple classical eigenvalue problem related to the un-damped case of a general dynamic system with \( N \) degrees of freedom (DOF’s)

\[
\mathbf{M}^{-1}\mathbf{Kb} = \omega^2 \mathbf{b}
\]

(1)

where \( \mathbf{M} \) is the mass matrix, \( \mathbf{K} \) is the stiffness matrix, \( \omega^2 \) is one of the positive and real eigenvalues. The natural frequencies and the mode shapes are found by the eigenvalue decomposition \( \mathbf{M}^{-1}\mathbf{K} = \mathbf{B}[\omega^2_n]^{-1} \mathbf{B}^{-1} \) where \( \mathbf{B} = [\mathbf{b}_n] \) is a matrix of eigenvectors and \( [\omega^2_n] \) is a diagonal matrix holding the eigenvalues.

The well-known orthogonality properties of the mode shapes are easily verified writing Eq. (1) for two different modes with the natural frequencies \( \omega_n, \omega_m \) and the mode shapes \( \mathbf{b}_n, \mathbf{b}_m \). Multiplying each of the equations with the other transposed mode shape from the left, subtracting the two equations and using that \( \mathbf{b}_n^T\mathbf{Kb}_m = \mathbf{b}_m^T\mathbf{Kb}_n \) we get
Thus we conclude that if the two eigenvalues are different, the inner product \( b^T_n M b_m = 0 \). This leads directly to the well-known orthogonality equations \( B^T M B = [m_i] \) and \( B^T K B = [k_i] \), where the diagonal matrices \([m_i]\) and \([k_i]\) holds the modal masses and the modal stiffness’s respectively.

Let us say that we consider a case where the two eigenfrequencies \( \omega_1, \omega_2 \) corresponding to the mode shapes \( b_1, b_2 \) are identical, thus \( \omega_1 = \omega_2 = \omega \). From Eq. (2) we see that the orthogonality between the two mode shapes \( b_1, b_2 \) is no longer assured, as the condition given by eq. (2) is always satisfied when \( \omega_1 - \omega_2 = 0 \). However, any linear combination of the two mode shapes \( b = t_1 b_1 + t_2 b_2 \) is also an eigenvector because from the eigenvalue problem (1) we have that \( M^{-1} K (t_1 b_1 + t_2 b_2) = \omega^2 b \).

This means that in the case of repeated poles the individual mode shapes does not make much sense as individual quantities. They can be rotated in the subspace, and thus, the subspace defined by the two mode shapes is the only important quantity defined by the two mode shapes.

It is to be expected that similar properties are related to the case of closely spaced modes and we will study this in more detail in the following.

3. SENSITIVITY EQUATIONS

Now let us consider the changes of the mode shapes and the natural frequency due to small changes of the system. The original treatment of this problem is due to Fox and Kapoor [4] and Nelson [5], but we will use the results from Heylen et al. [6]. In this text it is shown that the sensitivity of the mode shapes to changes of a parameter \( u \) in the dynamic system is

\[
\frac{\partial b_i}{\partial u} = -\frac{1}{2m_i} b_i^T \frac{\partial M}{\partial u} b_i + \sum_{r=1, r \neq i}^N \left( \frac{1}{\omega_i^2 - \omega_r^2} \right) b_r^T \left( -\omega_i^2 \frac{\partial M}{\partial u} + \frac{\partial K}{\partial u} \right) b_i \tag{3}
\]

where \( N \) is the number of modes in the model. Now considering the general, finite but small mass and stiffness changes \( \Delta M, \Delta K \), and because all terms of the form \( b_i^T \Delta M b_i \) are inner products (scalars) and therefore we can move the last vector in the products \( b_i^T \Delta M b_i, b_i \) to the front, we obtain the following approximate expression that is exact for the changes approaching zero

\[
\Delta b_i \approx -\frac{1}{2m_i} b_i b_i^T \Delta M b_i + \sum_{r=1}^N \frac{1}{\omega_i^2 - \omega_r^2} \left( -\omega_i^2 b_i b_i^T \Delta M b_i + b_i b_i^T \Delta K b_i \right) \tag{4}
\]

We can also obtain similar equations for the changes of the natural frequencies. Based on the results in Heylen et al. [6], these changes can be expressed as

\[
\Delta \omega_i = \frac{\omega_i^2}{2m_i} b_i b_i^T \left( -\Delta M + \frac{1}{\omega_i^2} \Delta K \right) b_i \tag{5}
\]

Now we consider a case where the two eigenfrequencies \( \omega_1, \omega_2 \) corresponding to the mode shapes \( b_1, b_2 \) are close, thus \( \omega_1 - \omega_2 = \Delta \omega \) and the frequency distance \( \Delta \omega \) is significantly smaller than all other frequency distances in the system, then we see from Eq. (4) that due to the weighting term given by \( 1/(\omega_1^2 - \omega_r^2) \), only the terms from the two closely spaced modes will significantly contribute to the change of the mode shapes of the two modes. For instance, considering a pure mass change, we get the mode shape change approximation for the mode shape \( b_1 \).
\[ \Delta b_i \simeq \frac{1}{2m_1} b_i b_i^T \Delta M b_i - \frac{1}{(\omega_1^2 - \omega_2^2)m_2} \omega_2^2 b_i b_i^T \Delta M b_i \] (6)

And we see that the mode shape change is a linear combination of the initial mode shape vectors. This means that the perturbed set of mode shapes approximately is a rotation of the existing mode shapes in their common sub-space. It can be shown that the mode shape rotations in the subspace can be expressed by one common rotation angle. This will be further investigated in the following sections using the principle of local correspondence (LC), Brincker et al. [7].

4. **ROTATION ANGLE OF MODERATELY SPACED MODES**

We are studying the case of closely spaced modes using the LC principle considering two systems, the unperturbed system (before perturbation) with mode shapes \( B \), and the perturbed system (after perturbation) with mode shapes \( A \). From the theory of structural modification, Sestieri et al. [8], we know that for a full set of mode shapes, the following equation exist for any finite changes of the mass and stiffness matrices \( \Delta M, \Delta K \)

\[ A = BT \] (7)

This equation express the fact that the mode shapes of the modified system can be expressed as a linear combination of the mode shapes from the initial system, the linear combination is defined by the transformation matrix \( T \). For a truncated set of mode shapes, it can be shown based on the sensitivity equation given by Eq. (4) noting that the first part \( b_i b_i^T \) of the matrix product \( b_i b_i^T \Delta M b_i \) is an outer product and following similar arguments as we just did for two closely spaced modes in the preceding section, Brincker et al. [7], that the same equation still holds as an approximation

\[ A \simeq BT \] (8)

where the LC principle states that if the modes are sorted according to frequency in the mode shape matrices, then the transformation matrix \( T \) is a sparse matrix, i.e. only a few elements around the diagonal needs to be taken into account. The LC principle and the sensitivity equations also provide analytical solutions for the matrix elements in the transformation matrix \( T \).

In Brincker and Lopez-Aenlle [9] it is shown that using the LC principle on the case of two closely spaced modes \( \omega_1 = \omega, \omega_2 = \omega + \Delta \omega \) leads to the following solution for the (transposed) of the transformation given by Eq. (8)

\[ \begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} = R \begin{bmatrix} b_1^T \\ b_2^T \end{bmatrix} \] (9)

Where the matrix \( R \) given by

\[ R = \begin{bmatrix} 1 & \theta \\ -\theta & 1 \end{bmatrix} \] (10)
is a rotation matrix. In general the definition of a rotation matrix defined by the counterclockwise rotation angle $\theta$ in a plane is

$$R = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

(11)

However, for small rotations $\cos \theta \approx 1$, $\sin \theta \approx \theta$ and the general rotation matrix reduce to the one given by to Eq. (10). The rotation angle is given by

$$\theta \approx \frac{\omega}{2\Delta \omega} \frac{\Delta m - \Delta k}{m}$$

(12)

where $m$ is the modal mass (assumed to be the same for the two mode shapes) and the scalars $\Delta m$, $\Delta k$ describing the changes of mass and stiffness are given by the inner products

$$\Delta m = b_1^T \Delta M b_2 = b_2^T \Delta M b_1$$

$$\Delta k = \frac{1}{\omega^2} b_1^T \Delta K b_2 = \frac{1}{\omega^2} b_2^T \Delta K b_1$$

(13)

As we have indicated previously, when the frequency distance $\Delta \omega$ gets close to zero, then the sensitivity of the mode shapes increase significantly. In fact, according to Eq. (12), the sensitivity approaches infinity for $\Delta \omega \to 0$. In the paper by Brincker and Lopez-Aenlle [9] the following approximation has been used $\omega^2 / (\omega_2^2 - \omega_1^2) \approx \omega / (2\Delta \omega)$. If we do not use this approximation, the following slightly improved equation for the rotation angle can be obtained

$$\theta \approx \frac{\omega^2}{\omega_2^2 - \omega_1^2} \frac{\Delta m - \Delta k}{m}$$

(14)

Still we see that the sensitivity for the rotation angle approaches infinity whenever frequency difference between the eigenvalues approaches zero. However it is interesting to note, that the rotation angle is found as the product of two terms, one term that we have just discussed

$$\frac{\omega^2}{\omega_2^2 - \omega_1^2} \approx \frac{\omega}{2\Delta \omega}$$

(15)

that describes the frequency spacing between the two modes, and the dimensionless term...
describing how the considered mass and stiffness changes influence the mode shapes of the considered set of closely spaced modes.

It should be noted that if the mass and stiffness change matrices are proportional to the initial mass and stiffness matrices then both terms \( \Delta m = 0 \), \( \Delta k = 0 \) and the rotation angle is zero. However, if this is not the case the quantity given by eq. (16) will reach a finite value, and the rotation angle is determined solely by the frequency ratio given by Eq. (14).

Further it should be noted that even though mass and/or stiffness changes are not proportional to the initial mass and stiffness matrices, the dimensionless term given by Eq. (16) might be zero leading to zero rotation. Popular speaking, a mass reduction might be canceled by a stiffness reduction with exactly the same effect on the rotation angle just with the opposite sign.

What can be concluded is that the actual sensitivity of the mode shapes depends on the actual mass and stiffness change matrices \( \Delta M \) and \( \Delta K \). Preliminary investigations (further investigations needed) indicate that if a random mass change matrix is used that will typically change the mass of the system with totally of the order of 1 %, then the corresponding dimensionless parameter \( \Delta m / m \) is approximately one order of magnitude smaller, that is around 0.1 %. We can use that to suggest a rough estimate for what it means that modes are closely spaced. If we say that this should result in maximum rotation angles of about 0.5 (which corresponds to about 28 deg or a MAC value that change from unity to 0.77), then from Eq. (12) we have the condition

\[
\frac{\omega}{2\Delta \omega} > 0.5; \quad \Rightarrow \quad \frac{\omega}{\Delta \omega} > 1000; \quad \frac{\Delta \omega}{\omega} < 0.001
\]

In practice this means that if \( \omega / \Delta \omega > 1000 \) then it must be expected that random mass distributions with a total mass change of 1 % will make it difficult to achieve higher MAC values between a test and a modal than about 0.80. This illustrates the need for better measures than the MAC (or any other one-by-one mode shape correlator) in the case of closely spaced modes.

If we have \( \omega / \Delta \omega < 100 \), then the rotation angle will for the same mass change be smaller than 0.05 (which corresponds to about 2.8 deg or a MAC value that change from unity to 0.998). This means that if the modes fulfil this condition, the changes due to random mass change of a total magnitude of 1 % is not expected to have any influence on the MAC value between the perturbed and the unperturbed set of modes.

### 5. Rotation Angle of Very Closely Spaced Modes

The solution derived in the preceding section is only useful for “moderate closeness” of the two closely spaced eigenvalues because the solution for the rotation angle by Eqs. (12/14) does not make sense when \( \omega / \Delta \omega \to \infty \).

Therefore we are now looking at a case with more closely spaced eigenvalues than in the preceding section. After a perturbation this might cause the two “very closely spaced eigenvalues” to separate more, but we assume that in the initial state the two modes are so closely spaced that we are in the same situation as for the repeated eigenvalues case analyzed in the introduction where the eigenvectors are indeterminate as individual quantities and only define the subspace. Therefore, we could also as the initial vectors use the vectors \( \mathbf{a}_1, \mathbf{a}_2 \) that constitute the perturbed set, because also this set is in the subspace of the initial vectors. Then from Eq. (9) we get

\[
\frac{\Delta m - \Delta k}{m}
\]
Thus for the new set of modes \( A = [a_1, a_2] \) the rotation angle \( \theta \) must vanish. In Brincker and Lopez-Aenlle [9] it is shown that this leads to the eigenvalue problem

\[
B^T \left( -\Delta M + \frac{1}{\omega^2} \Delta K \right) B = T D T^{-1}
\]  

(19)

Where \( B \) is the mode shape matrix holding the two initial mode shapes \( b_1, b_2 \), \( T \) is the transformation matrix defined in Eqs. (7/8) holding the eigenvector to the matrix defined as the right hand side of Eq. 19), and \( D \) is a diagonal matrix holding the eigenvalues. The eigenvalues are related to the frequency shift defined by Eq. (5) by

\[
D = \frac{2m}{\omega} \begin{bmatrix} \Delta \omega_1 & 0 \\ 0 & \Delta \omega_2 \end{bmatrix}
\]  

(20)

In this case the rotation angle is found noting that the transformation matrix \( T \) is just the rotation matrix \( R \) transposed \( T = R^T \), thus, the rotation angle is found as the off-diagonal element of the transformation matrix.

In Figure 2 is shown some results of an analysis of a 3 degree of freedom system from Brincker and Lopez-Aenlle [9] where the rotation angle is calculated for a certain mass change distribution. The figure shows the two solutions for the rotation angle that we have obtained above; Eq. (12) that is valid for moderately spaced modes, and Eq. (19) that is valid for large rotations. As we see from the figure, in this example, the mass change leads to insignificant rotations if \( \omega/\Delta \omega < 1000 \), in the region up to \( \omega/\Delta \omega = 10^4 \), the approximation given by Eq. (12) is quite good, but overestimates the rotations for larger values of \( \omega/\Delta \omega \). We see that the eigenvalue solution given by Eq. (19) provides an upper bound for the rotation. We also see that using the smallest rotation angle of the two solutions will in some cases overestimate the rotation angle by a factor of two.

6. TEST-MODEL CORRELATION

In the preceding sections, we have provided analytical solutions to support the fact that for a long time has been generally accepted in modelling of dynamical systems and in modal testing:

- The mode shapes of closely spaced modes are highly sensitive to small mass and stiffness perturbations of the system,
- The mode shape changes are mainly due to rotations of the mode shapes in the initial subspace

This means that the classical one-by-one mode shape correlation between test and model cannot be used without facing significant problems with low correlation values even though the model describes the physics of the test specimen quite well.

In the following the unperturbed set of modes with the mode shape matrix \( B = [b_1, b_2] \) can be thought of as the finite element (FE) model, and the perturbed set of modes with the mode shape matrix \( A = [a_1, a_2] \) can be thought of as the test results.

We will now discuss what we do with the problem, that for a nearly perfect FE model we might have bad correlation between the two sets of modes on a one-by-one mode basis.
Figure 2. Rotation angle for a 3 degree of freedom system considering a certain mass change. After Brincker and Lopez-Aenlle [9].

One way to deal with this problem is to use the freedom of the mode shapes to rotate (that causes the problem) to rotate the mode shapes back (to solve the problem). Let us say that the modes in the FE model are more closely spaced than in the test. In this case it is natural to rotate the mode shapes in the FE model to make them correlate in the best possible way with the test mode shapes. This can be done by trying to write the test mode shape $a_1^*$ as a linear combination of the two FE mode shapes

$$ a_1 \approx Bt $$

(21)

Where $t$ is the unknown linear combination vector with two elements, one for each mode. We can find an approximate solution $\hat{t}$ to this equation by

$$ \hat{t} = B^+a_1^* $$

(22)

Where $B^+$ is the pseudo inverse of $B$. We can then define a new FE mode shape

$$ c_1 = B\hat{t} $$

(23)

and we can now correlate the new FE mode shape vector $c_1$ with the experimental vector $a_1$. This procedure has recently been proposed by Skafte et al. [10] and is illustrated in Figure 3. In Skafte et al. it shown that the error that is introduced by rotating the FE models to obtain the best possible correlation with the experimental mode shapes can be considered as not significant for practical purposes. The procedure is closely related to the procedure proposed by D’Ambrogio and Fregolent [11].
The mode shapes of two close spaced modes from a FE model (B) can be correlated with the corresponding mode shapes from a test (A) by rotating the FE mode shapes in their subspace to obtain a new set of FE mode shapes (C) with maximum correlation with the experimental mode shapes. After Skafte et al. [10].

From Figure 3 it is natural to conclude, that since the largest possible angle between the new FE mode shape vectors and the experimental mode shape vectors is equal to the angle between the two planes indicated in Figure 3, then it is enough to calculate the angle between the planes A and B. In general this corresponds to estimating the subspace angle between the two subspaces $A$ and $B$.

In general there exist as many subspace angles as we have vectors in the subspaces. In this case where we are studying two closely spaced modes, two angles are present. In the case of a 3D vector (which is the case for the plots shown in Figure 3), the two angles are the angle between the plane (as mentioned earlier) and then the angle in the intersection line between the two planes, which is zero. For vectors with higher dimension than 3, one can consider both angles because in this case normally they will both be different from zero.

If just the largest subspace angle is to be used, then a Matlab commend like \( \text{theta} = \text{subspace}(A,B) \) can be used to find the largest subspace angle. If both angles are needed, then they can be calculated as prescribed by Golub and Van Loan, [12].

Finally it should be noted that any generalized angle $\theta$ (between mode shape vectors and/or subspace angles) can be expressed as a MAC value, see Allemang et al. [13], by the simple formula

$$MAC = \cos^2(\theta)$$  

\[24\]

7. CONCLUSIONS

We have studied the case of closely spaced modes considering

- The case of repeated poles where we revisit the well-known fact that the two mode shapes do not exist as individual quantities and only the subspace spanned by the mode shapes makes sense
- The case of moderately spaced modes where it is shown that small perturbations mainly leads to rotations of the mode shapes in their initial subspace; an analytical solution for the rotation angle is given that that turns out to be proportional to the frequency ratio $\omega / \Delta \omega$
- The case of very closely spaced modes which leads to an eigenvalue problem that provides a solution for the rotation angle that is independent of the frequency ratio $\omega / \Delta \omega$ and serves as an upper bound for the rotation angle.
We have then used these results to discuss what it means that modes are closely spaced and how we can treat the problem of over sensitive mode shapes when performing test-model correlation. It is proposed that modes are considered to be closely spaced whenever the frequency ratio \( \omega_1/\Delta \omega > 1000 \). In this case it must be expected, that due to the inherent sensitivity created by the closeness of the modes, it is difficult to correlate mode shapes on a one-by-one basis with higher MAC value than about 0.80. Finally in order to deal with the problem of low correlation, it is proposed to rotate the FE mode shapes in their subspace to maximize the correlation with the experimental mode shapes, or to use the maximum subspace angle between the two mode shape subspaces.

REFERENCES


