STOCHASTIC MODAL IDENTIFICATION IN THE PRESENCE OF HARMONIC EXCITATIONS

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ABSTRACT
Structures are often subjected to harmonic excitations in addition to random ambient noise. Under such a mixed loading situation, applying classical operational modal analysis (OMA) techniques is likely to encounter difficulties for correctly identifying structural modal parameters. Generally, harmonic components are potentially mistaken for being structural modes or might bias the estimation of structural modes. As various OMA time-domain methods are modeled differently with distinct solution algorithms, adopting these methods would have different numerical conditioning and stability. The objective of this study is to identify the most effective time-domain method for stochastic modal identification in the presence of harmonic excitations. In the numerical study, dynamical responses of a 5-DOF mass-damper-spring system are simulated. The system is to be excited by the harmonic loading, together with random white noise loading. The projection-driven stochastic subspace identification (SSI) method is found to be superior to the covariance-driven instrumental variable (IV) and stochastic realization algorithm (S-RA) methods. From the numerical results of the scenario that the harmonic frequency is near to a modal frequency, the estimated modal parameters from the SRA method are highly perturbed and its stability diagrams are prone to produce stable spurious poles. Meanwhile, employing the stability diagrams from the IV method totally loses the ability to identify the close modal frequency. In contrast, the SSI method could accurately identify the harmonic frequency and the structural modal information. Because of its good numerical conditioning and robust numerical stability, the SSI method is a promising tool for the operational modal analysis in the presence of harmonic excitations, even without any “modification”.

Keywords: Stochastic modal identification, Instrumental Variable, Stochastic Realization Algorithm, Stochastic Subspace Identification
1. **INTRODUCTION**

Operational modal analysis (OMA) methods always assume that the input is a zero-mean Gaussian white noise random process. In practice, however, structures are often subjected to harmonic excitations in addition to random ambient noise. Under such a mixed loading situation, applying classical OMA techniques is likely to encounter difficulties for correctly identifying structural modal parameters. Generally, harmonic components are potentially mistaken for being structural modes or might bias the estimation of structural modes [1].

In order to extract the true structural modes under the mixed loading situation, a number of techniques have been proposed. An intuitive approach is to firstly eliminate harmonic components from the measured signals, which is usually accomplished by a numerical filter or by using the random decrement technique [2]. However, this approach is practically effective only when the harmonic frequencies are well separated from the structural frequencies of the system. Another approach of dealing with the harmonics is simply treating them as virtual modes with zero damping. Unfortunately, when harmonic frequencies are close to natural frequencies of the system, the identified structural frequencies and damping ratios from several experimental modal analysis methods have been shown to be highly perturbed from the true values. To address this problem, several modified versions of the traditional modal analysis methods have been proposed [3, 4]. One branch of the modified methods is to implement algorithms that force the identified system matrices to have the exact eigensolutions associated with the harmonics. This modification allows modal modes to be computed accurately even if the harmonic frequencies are close to the structural frequencies. However, they are valid only if the harmonic frequencies are exactly known a priori.

As various OMA time-domain methods are modeled differently with distinct solution algorithms, adopting these methods would have different numerical conditioning and stability [5]. The objective of this study is to identify the most effective OMA method under the situation that the excitation is the combination of a random white noise and harmonic components with unknown frequencies. The projection-driven stochastic subspace identification (SSI) method is particularly advocated for this study because of its superiority on numerical conditioning (using first-order state-space stochastic model) and stability (using a robust projection algorithm). In the numerical study, dynamical responses of a 5-DOF mass-damper-spring system are simulated. The advantage of employing a simulated system is that the estimation can be evaluated by the true modal information. The system is to be excited by the harmonic loading, together with random white noise loading. Two distinct scenarios are investigated: (1) the harmonic frequency being far from all modal frequencies, and (2) the harmonic frequency being near to a modal frequency. For making a comparison among various OMA methods, the covariance-driven instrumental variable (IV) method and stochastic realization algorithm (SRA) method, in addition to the projection-driven SSI method, are also carried out.

2. **STOCHASTIC MODELS FOR MODAL ANALYSIS**

All time-domain methods for experimental modal analysis (EMA) begins with a mathematical model, either a high-order matrix polynomial model or a first-order state-space model. For OMA, the high-order model is commonly known as the ARMA model [6], which can be formulated as:

\[
\sum_{j=0}^{n_\alpha} \alpha_j y_{k-j} = \sum_{j=0}^{n_\gamma} \gamma_j e_{k-j}
\]

where \( y_{k-j} \in \mathbb{R}^{N_o} \) is the output vector and \( e_{k-j} \in \mathbb{R}^{N_o} \) is a white noise vector sequence, in which \( N_o \) is the number of outputs. The left-hand side is called the Auto-Regressive (AR) part and the right-hand side the Moving-Average (MA) part, hence the name of the model. The matrices \( \alpha_j \in \mathbb{R}^{N_o \times N_o} \) are the AR matrix parameters; and the matrices \( \gamma_j \in \mathbb{R}^{N_o \times N_o} \) are the MA matrix parameters. It should be reminded
that an ARMA model that is deduced from a state-space model has the same AR order $n_\alpha$ as MA order $n_\gamma$. The first-order stochastic state-space model can be written as [7]:

$$
x_{k+1} = Ax_k + w_k \\
y_k = Cx_k + v_k
$$

(2)

where $w_k \in \mathbb{R}^n$ is the process noise with $n$ the number of model order, and $v_k \in \mathbb{R}^{N_o}$ is the measurement noise. Both noise vectors are unmeasurable vector signals assumed to be zero mean, white and with covariance matrices:

$$
E \left[ \left( \begin{array}{c} w_p^T \\ v_p^T \end{array} \right) \left( \begin{array}{c} w_q^T \\ v_q^T \end{array} \right) \right] = \left[ \begin{array}{cc} Q & S \\ S^T & R \end{array} \right] \delta_{pq}, \text{ where } E \text{ is the expected value operator and } \delta_{pq} \text{ is the Kronecker delta.}
$$

3. STOCHASTIC MODAL ANALYSIS METHODS

This section reviews the fundamental algorithms of the three stochastic modal analysis methods to be applied in this paper: covariance-driven IV and SRA methods, and projection-driven SSI method. Particular effort is to highlight the similarity and difference among them.

3.1. Instrumental Variable (IV) Method

The IV method, based on ARMA model, extracts the maximum information from the data by leaving residuals $e_k$ uncorrelated with past data. The advantage of the IV method is that it identifies only the AR parameters, while the underlying model structure still is an ARMA model. The basic IV equation can be written in terms of the output covariances [6]:

$$
\sum_{j=0}^{P} \alpha_j \Lambda_{p+i-j} = 0, \text{ for } i > 0
$$

(3)

where $\Lambda_i = E[y_{k+i}y_k^T]$ are the output covariances. By applying the output covariances and writing down the equation for all available time lags, the AR parameters can be estimated by solving the resulting over-determined set of equations in a least squares sense. Finally, the eigenvalues and the observed mode shapes are obtained from the eigenvalue decomposition of the companion matrix of the AR coefficients [8]. It is clear that the final formulation of the IV method in Eq. 3 corresponds to the mathematical model of the deterministic polyreference complex exponential (PRCE) method after substituting impulse responses by output covariances [7]. Implementing IV method is actually the same as utilizing the Natural Excitation Technique (NExT) and then employing the PRCE method for stochastic modal analysis, which was applied in Ref.[3].

3.2. Stochastic Realization Algorithm (SRA) Method

Referring to Eq. 2, one assumes the response stochastic process to be stationary with $E[x_k] = 0$ and $E[x_kx_k^T] = \Sigma$, where the state covariance matrix $\Sigma$ is independent of the time $k$. This implies that $A$ is a stable matrix. Since $w_k$ and $v_k$ are zero mean white noise vector sequences, independent of $x_k$, one thus has $E[x_kw_k^T] = 0$ and $E[x_kv_k^T] = 0$. Then one finds the Lyapunov equation for the state covariance matrix: $\Sigma = A\Sigma A^T + Q$. Defining the covariance matrix between $x_{k+1}$ and $y_k$ as $G = E[x_{k+1}y_k^T] \in \mathbb{R}^{n \times N_o}$, one can obtain: $\Lambda_0 = C\Sigma C^T + R$, $G = A\Sigma C^T + S$ and

$$
\Lambda_i = CA^{i-1}G, \quad i = 1, 2, \ldots
$$

(4)
The classical algorithm of SRA method is to arrange the output covariance matrices $\Lambda_i$ in a Toeplitz matrix form
\[
C_i = \begin{bmatrix}
\Lambda_i & \Lambda_{i-1} & \cdots & \Lambda_1 \\
\Lambda_{i+1} & \Lambda_i & \cdots & \Lambda_2 \\
\cdots & \cdots & \cdots & \cdots \\
\Lambda_{2i-1} & \Lambda_{2i-2} & \cdots & \Lambda_i 
\end{bmatrix} = \Gamma_i \Delta_i
\] (5)
where $\Gamma_i \in \mathbb{R}^{N_o \times n}$ is the extended observability matrix and $\Delta_i$ is the reversed extended stochastic controllability matrix. They can be written as:
\[
\Gamma_i = [ C \ CA \ \cdots \ CA^{i-1}]^T, \quad \Delta_i = [ A^{i-1} G \ \cdots \ \AG \ G ]
\] (6)

There are several ways to get a minimum realization for the system matrix $A$ [8]. One way is to use only the extended observability matrix $\Gamma_i$ (see Eq. 5). Let the singular value decomposition of $C_i$ to be expressed as:
\[
C_i = ( U_1 \ U_2 ) \left( \begin{array}{cc}
S_1 & 0 \\
0 & 0
\end{array} \right) \left( \begin{array}{c}
V_1^T \\
V_2^T
\end{array} \right) = U_1 S_1 V_1^T
\] (7)

In practice, the order of the system, $n$, is set equal to the number of singular values in Eq. 7 above a preset threshold. One possible realization $\Gamma_i$ is $\Gamma_i = U_1 S_1^{1/2}$. The matrix $A$ can now be determined from the extended observability matrix by making use of the shift structure of the matrix $\Gamma_i$. Denote $\Gamma_i$ as the extended observability matrix $\Gamma_i$ without the first $l$ rows, and $\Gamma_i^s$ the extended observability matrix $\Gamma_i^s$ without the last $l$ rows. Observing Eq. 6 one can easily show $\Gamma_i^s = \sum A$, which suggests that an estimate for $A$ can be computed as:
\[
\hat{A} = \Gamma_i^{sT} \Gamma_i
\] (8)
where the superscript "\text{\textsuperscript{+}}" denotes the Moore-Penrose pseudo-inverse of a matrix. The estimated $C$ could be obtained from the first block of $\Gamma_i$.

Because flipping left to right a Toeplitz matrix (see Eq. 5) yields a Hankel matrix, a realization of $A$ can also be estimated from the information of covariance matrices by adopting the same steps of the Eigensystem Realization Algorithm (ERA) [7]. Implementing SRA method is actually the same as utilizing the Natural Excitation Technique (NExT) and then employing the ERA method for stochastic modal analysis, which was applied in Ref. [4].

3.3. Stochastic Subspace Identification (SSI) Method

The stochastic subspace identification method (SSI) avoids the computation of covariances between the outputs. It is replaced by projecting the row space of future outputs into the row space of past outputs. A detailed derivation of the stochastic subspace identification algorithm is given in [9].

The output measurements $y_k \in \mathbb{R}^{N_o}$, are gathered in a block Hankel matrix $H_{0[2i-1]}$ with $2i$ block rows and $j$ columns, where $j = s - 2i + 2$:
\[
H_{0[2i-1]} = \frac{1}{\sqrt{j}} \begin{bmatrix}
y_0 & y_1 & \cdots & y_{j-1} \\
y_1 & y_2 & \cdots & y_j \\
\cdots & \cdots & \cdots & \cdots \\
y_{i-1} & y_i & \cdots & y_{i+j-2} \\
y_i & y_{i+1} & \cdots & y_{i+j-1} \\
y_{i+1} & y_{i+2} & \cdots & y_{i+j} \\
\cdots & \cdots & \cdots & \cdots \\
y_{2i-1} & y_{2i} & \cdots & y_{2i+j-2}
\end{bmatrix} = \begin{bmatrix}
\mathbf{Y}_p \\
\mathbf{Y}_f
\end{bmatrix}
\] (9)
The matrices \( Y_p \) and \( Y_f \) are defined by splitting \( H_{0|2i−1} \) into two parts of \( i \) block rows, in which the subscripts \( p \) and \( f \) stand for past and future. The projection of the row space of the future outputs \( Y_f \) into the row space of the past outputs \( Y_p \) is, denoted by \( Y_f / Y_p \), can be computed by:

\[
\mathcal{P}_i = Y_f / Y_p = Y_f Y_p^T (Y_p Y_p^T)^+ Y_p
\]  

(10)

However, a more accurate way to obtain \( \mathcal{P}_i \) numerically is via the QR-factorization of \( H_{0|2i−1} \), instead of computing the above formula directly.

In the theoretical derivations of the stochastic subspace identification, one assumes that the number of measurements goes to infinity. Assuming \( j \to \infty \), the main theorem of stochastic subspace identification states that the projection \( \mathcal{P}_i \) is equal to the product of the observability matrix \( \Gamma_i \) and the Kalman filter state sequence \( \hat{X}_i \) [9]:

\[
\mathcal{P}_i = \Gamma_i \hat{X}_i
\]  

(11)

where the forward Kalman filter state sequence is \( \hat{X}_i = ( \hat{x}_i \hat{x}_{i+1} \cdots \hat{x}_{i+j-2} \hat{x}_{i+j-1} ) \in \mathbb{R}^{n \times j} \). This theorem can algebraically be summarized as follows: (1) rank of \( \mathcal{P}_i = n \), (2) row space of \( \mathcal{P}_i = \) row space of \( \Gamma_i \), and (3) column space of \( \mathcal{P}_i = \) column space \( \hat{X}_i \). This summary is the essence of why the projection-based algorithm has been called a subspace algorithm, as it retrieves system related matrices as subspaces of projected data matrices. Several approaches can be utilized to estimate \( A \) and \( C \). One of them is to use the extended observability matrix alone (see Eq. [8]). Another approach uses the Kalman filter state sequence together with the extended observability matrix [9].

### 3.4. Numerical conditioning and stability

As various OMA methods are modeled differently with distinct solution algorithms, implementing these methods would have different conditioning and stability. Numerical conditioning pertains to the perturbation behavior of a mathematical problem (model) itself and stability pertains to the perturbation behavior of an algorithm used to solve that problem on a computer.

Comparing the numerical conditioning of various modal analysis methods, it has been theoretically illustrated that methods based on first-order state-space models are likely to be better conditioned than those based on high-order polynomial models [5]. While estimating the modal frequencies and damping values, high-order model methods always need to obtain the coefficients of the high-order polynomial model first and then find the roots of the polynomial. In contrast, methods using a state-space model need to obtain the state matrix associated with its first-order matrix model and then compute the eigenvalues of the state matrix. It is a well established knowledge that the eigenvalue problem is usually well-conditioned, but the problem of finding the roots from the polynomial coefficients is often ill-conditioned [10]. Because the eigenvalues of the companion matrix of the polynomial coincide with the roots of the polynomial, one can use any eigenvalue algorithm to find the roots of the polynomial. But finding polynomial roots from the companion matrix of the polynomial would not change the condition of the original polynomial root-finding problem. On the other hand, it is a bad strategy to compute eigenvalues of a matrix by first calculating the coefficients of the matrix’s characteristic polynomial and then finding its roots, since using the coefficients as an intermediate step may introduce an extreme ill-conditioning even if the underlying eigenvalue problem is well-conditioned.

The SRA and SSI methods are based on the same first-order stochastic state-space model and the implementation of them is similar. It consists of computing the SVD of the kernel matrix, truncating the SVD to the model order \( n \), estimating the observability matrix and controllability matrix (or Kalman filter state sequence for SSI) by splitting the SVD in two parts, and finally estimating system matrices from the observability matrix. It has been proven that the two methods are theoretically equal on estimating the system matrices when \( j \to \infty \). In numerical implementation, however, there are many factors that might cause discrepancy between the SSI and SRA methods, such as the selection of finite values for \( i \) and \( j \),
and the numerical procedures to get the Toeplitz matrix $C_i$ and the projection $P_i$. A numerically favorable approach to compute $P_i$ is using the QR-factorization of the Hankel matrix $H_{0/2i-1}$. Furthermore, the SVD of the projection is obtained with only the R-factor being needed in the algorithm [9]. The most accurate way to compute the output-covariance Toeplitz matrix is by using a time domain approach. Throughout this paper, Matlab function “xcorr” with the unbiased option is employed to compute the output covariances [11].

4. NUMERICAL STUDY

A 5-DOF mass-spring-dashpot system (see Fig. 1) is adopted for numerical study. The system is with uniform mass, stiffness and damping coefficients: $m_n = 50$ kg, $k_n = 2.9 \times 10^7$ N/m, and $c_n = 1000$ N·s/m, for $n = 1, \cdots, 5$. The coordinates of the 5-DOF model are denoted by $x_n$, with $x_1$ at the fixed end and $x_5$ at the free end. Performing the eigen analysis of this system, the five modal frequencies are 34.4996, 100.7039, 158.7498, 203.9347 and 232.5980 Hz. The corresponding modal damping ratios are 0.3737%, 1.0909%, 1.7197%, 2.2092% and 2.5198%, respectively. By these “true” analytical modal values, the estimation can be evaluated. For investigating the stochastic modal analysis in the presence of harmonic excitation, the input loading to the system is synthesized by a random noise and a harmonic component. The expression for the harmonic component $s_k$ is $s_k = A_h \sin(2\pi f_h t_k + \phi_h)$, $k = 1, \cdots, K$, where $A_h$, $f_h$ and $\phi_h$ are the amplitude, frequency and phase angle of the harmonic component, respectively. Moreover, the output noise is simulated by adding Gaussian white noise to the output-noise-free response from the system. The level of the additive output noise is quantified by a stated percentage, defined as the ratio of the standard deviation of the white noise to that of the clean response.

Specifically in this paper, the dynamic response at coordinate $x_1$ has been obtained by employing Matlab build-in function “lsim”, which computes the response of a linear time-invariant (LTI) system subjected to an arbitrary input [11]. The input is a discrete signal at coordinate $x_1$ with the sampling interval $\Delta t=0.002s$ for 8192 time steps. Embedded in the input are a zero-mean Gaussian white noise with standard deviation, and a harmonic component with $A_h=2$ and $\phi_h=0$. The simulated response signal is further contaminated by a 5 percent measurement noise. In the following, two distinct scenarios are investigated: (1) the harmonic frequency being far from all modal frequencies, and (2) the harmonic frequency being near to a modal frequency.

4.1. Harmonic excitation with frequency far from all modal frequencies ($f_h = 120$ Hz)

When the frequency of the harmonic component is $f_h=120$Hz, it is far from all the structural modal frequencies. Traditionally, the first step to employ a time-domain modal analysis method is determining the model order, which could be achieved by analyzing the normalized singular values of the projection matrix $P_i$ or the covariance matrix $C_i$. A conventional way to choose the model order is to find a significant gap of the normalized singular values, but the choice is usually subjective. In this study, with the total length of the signal be 8192 data points, the Hankel matrix for the SSI method has been constructed with $i=250$ and $j=7693$. The selection of this size is not arbitrary, but a careful choice based on multiple trials with different $i$ and $j$ (noting $2i + j - 1=8192$). It was found that the selection of $i=250$ and $j=7693$ could identify more modes with better estimated values. When $i=250$, the size of
The projection $P_i$ for the SSI method is $250 \times 7693$, and the Toeplitz matrix $C_i$ for the SRA method is $250 \times 250$. The curves of the normalized singular values for both $P_i$ and $C_i$ are shown in Fig. 2. Observing the curve of $P_i$ and $C_i$, it is found the model order for SSI could be easily set as 10, but it is hard to ascertain the model order for covariance-driven methods is 4 or 22. Taking model order as 10, the estimated frequencies from SSI method are 34.5527Hz, 100.5796Hz, 119.9992Hz, 158.9933Hz and 202.1657Hz. The corresponding damping ratios are 0.3340%, 1.0481%, 0.0001%, 1.5623% and 4.0850%, respectively. Obviously, four structural modes are covered in the estimation. The estimated damping ratio 0.0001% is very close to zero, which can prove the corresponding frequency 119.9992Hz is from a harmonic excitation rather than a real structural mode.

![Figure 2: Normalized singular values of $P_i \in \mathbb{R}^{250 \times 7693}$ and $C_i \in \mathbb{R}^{250 \times 250}$ while $f_h=120$Hz](image)

Nowadays, one is less interested in good model order determination, but rather in the modal parameters extracted from running models with different orders. Past experience with real data showed that it was better to over-specify the model order and to eliminate spurious numerical poles afterwards. This could be done by constructing stability diagrams. With the model order varying from 3 to 30, the stability diagram obtained from implementing the SSI, SRA and IV methods are shown in Figs. 3-5, in which the poles are labeled as stable when they are within the limitations of 1% difference in frequency and 5% in damping ratio between two consecutive model orders, namely: $|f^{(n)} - f^{(n+1)}|/f^{(n)} < 1\%$ and $|\xi^{(n)} - \xi^{(n+1)}|/\xi^{(n)} < 5\%$, where $f^{(n)}$ and $\xi^{(n)}$ denote the estimated frequency and damping ratio with model order $n$. Throughout this paper, the same stability criteria are applied to all stability diagrams. The background vertical lines shown in these stability diagrams are corresponding to the true modal frequencies and the harmonic frequency. The background curve of the stabilization diagram is a Fourier spectrum, which is obtained from the DFT (discrete Fourier transform) of the sample signal, with the frequency resolution 0.0610Hz.

![Figure 3: Stability diagram from the SSI method while $f_h=120$Hz](image)
Figure 4: Stability diagram from the SRA method while $f_h=120\text{Hz}$

Figure 5: Stability diagram from the IV method while $f_h=120\text{Hz}$

The virtual mode associated with the harmonic frequency of 120Hz can be estimated from the stability diagrams by all methods. It can be estimated by the SSI method while the model order is 6. In contrast, the covariance-driven methods (SRA and IV) cannot identify it before the model order gets almost 22. This is in coincidence with the singular value curves associated with covariance matrix $C_i$ shown in Fig. 2 where there is a significant drop at the 23th singular value. To make a comparison, the estimated frequencies and damping ratios associated with the virtual mode by different model orders are listed in Tab. 1. This table indicates the estimated frequencies by SSI method are closest to the true frequency of the harmonic excitation, i.e. 120Hz. Moreover, the estimated damping ratios from the SSI method are almost zero (as small as $10^{-6}$), which reconfirms it is a virtual mode. In contrast, the estimated damping ratios from the IV and SRA methods are much larger than those from SSI method and consequently the virtual mode may potentially be mistaken as a structural mode. To sum up, the SSI method is the best one to identify the virtual mode associated with the harmonic excitation in this scenario.

One should keep in mind that the final goal of modal analysis is to identify the structural modal information. Fig. 3 demonstrates that the first four structural modes can be easily estimated by the SSI method. The outstanding performance of the SSI method is attributed to its first-order mathematical model and the robust QR decomposition algorithm to compute the projection. For the covariance-driven methods, IV method can identify four structural modes but SRA method only three. An obvious phenomenon associated with SRA method is that there are multiple spurious poles are stabilized around the strong modes, i.e. the second and third structural modes (see Fig. 4). The stabilization of spurious poles in the SRA method is attributed to the systematic error caused by the numerical error in the (square operation) computation of the covariance sequence and arrangement in one matrix $C_i$ (see Eq. 5). By stability diagrams, implementing the SRA method is likely to mistakenly identify these spurious (computational)
Table 1: Estimation for the virtual mode from harmonic excitation while \( f_h = 120 \text{Hz} \)

<table>
<thead>
<tr>
<th>Model order</th>
<th>Estimated frequencies</th>
<th>Estimated damping ratios</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>IV SRA SSI</td>
<td>IV SRA SSI</td>
</tr>
<tr>
<td>6</td>
<td>N/A N/A 120.1699</td>
<td>N/A N/A 0.0005%</td>
</tr>
<tr>
<td>8</td>
<td>N/A N/A 120.0063</td>
<td>N/A N/A 0.0001%</td>
</tr>
<tr>
<td>10</td>
<td>N/A N/A 119.9992</td>
<td>N/A N/A 0.0001%</td>
</tr>
<tr>
<td>12</td>
<td>N/A N/A 119.9986</td>
<td>N/A N/A 0.0001%</td>
</tr>
<tr>
<td>14</td>
<td>N/A N/A 119.9983</td>
<td>N/A N/A 0.0001%</td>
</tr>
<tr>
<td>16</td>
<td>N/A N/A 119.9986</td>
<td>N/A N/A 0.0001%</td>
</tr>
<tr>
<td>18</td>
<td>N/A N/A 119.9988</td>
<td>N/A N/A 0.0001%</td>
</tr>
<tr>
<td>20</td>
<td>118.0911 119.4005</td>
<td>119.9992 1.9891% 0.0420% 0.0001%</td>
</tr>
<tr>
<td>22</td>
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<td>119.9991 1.6930% 0.0374% 0.0001%</td>
</tr>
<tr>
<td>24</td>
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<td>119.9983 0.7138% 0.0341% 0.0001%</td>
</tr>
<tr>
<td>26</td>
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<td>119.9992 0.5469% 0.0397% 0.0000%</td>
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<tr>
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<td>119.9993 0.4228% 0.0298% 0.0001%</td>
</tr>
<tr>
<td>30</td>
<td>119.9320 119.9798</td>
<td>119.9999 0.2429% 0.0260% 0.0001%</td>
</tr>
</tbody>
</table>

poles to be true (physical) poles. The SRA method also loses the ability to identify the fourth mode, which is weak compared to other three structural modes. That is because, after the square operation, a small number will become even a smaller number. Though IV method also employs the covariance, it relies on a high-order ARMA model, with a changing model order. From Eq. 3, a series of difference equations are formulated and the equations associated with small time lags utilize the covariance matrices with high equality. However, the IV method has no mechanism to reduce noise, so its stability diagram is not clean (see Fig. 5).

4.2. Harmonic excitation with frequency far from all modal frequencies \( (f_h = 34.6000 \text{Hz}) \)

In this scenario, the frequency of the harmonic component is \( f_h = 34.6000 \text{Hz} \), which is very close the first structural modal frequency 34.4996Hz and thus provides a challenging problem. The curves of the normalized singular values for both \( P_i \in R^{250 \times 7693} \) and \( C_i \in R^{250 \times 250} \) while \( f_h = 34.6 \text{Hz} \) are shown in Fig. 6. The curves have a significant drop at the 9th and 7th singular value, respectively. Taking the model rank as 8, the estimated frequencies from the SSI method are 34.4778Hz, 34.5991Hz, 100.5821Hz and 159.1088Hz. The corresponding damping ratios are 0.4435%, 0.0002%, 0.8864%, 1.3622%, respectively. These certify the SSI method successfully recovers the 1st, 2nd, 3rd structural modes, and the virtual mode from the harmonic excitation. With model rank 6, the estimated frequencies from the IV method are 34.5956Hz, 102.8709Hz, and 189.1592Hz. The corresponding damping ratios are 0.0235%, 6.0713% and 20.6374%, respectively. Similarly, the estimated frequencies from the SRA method are 34.5166Hz, 34.6058Hz, 100.4720Hz, and
the corresponding damping ratios are 0.0432%, 0.0069%, and 1.3149%. These values demonstrate the IV method only roughly obtains the 2nd structural mode and the virtual mode, while the SRA method gets the 1st and 2nd structural modes and the virtual mode. It can be concluded after model order determination, the estimations from both the IV and SRA methods with model order \( n = 6 \) are not as good as those from the SSI method.

The stability diagrams from the SSI method, SRA method and IV method are displayed in Figs. 7-9. From close observation on the frequency range around the first structural mode and the harmonic excitation, it is found that only the poles from the SSI method correctly fit the true values. For the SRA method, spurious and unstable poles are generated around the two close frequencies with high perturbation. For the IV method, the poles associated with the harmonic excitation are formed but it totally loses the ability to identify the close structural mode. Both the IV and SRA methods have bad numerical performance for estimating the close modes. As a result, they are not recommended for the scenario that the harmonic frequency near to a structural frequency.

![Stability diagram from the SSI method while \( f_h = 34.6 \)Hz](image1.png)

**Figure 7:** Stability diagram from the SSI method while \( f_h = 34.6 \)Hz

![Stability diagram from the SRA method while \( f_h = 34.6 \)Hz](image2.png)

**Figure 8:** Stability diagram from the SRA method while \( f_h = 34.6 \)Hz

The estimated values associated with the virtual mode from the harmonic excitation \( f_h = 34.6 \)Hz are listed in Tab. 2. Once again, the estimated frequencies from the SSI method are closest to the true value 34.6Hz. Moreover, only the damping ratios from the SSI method are as small as 0.0002%, which are very close to zero and can prove they are not because of a structural mode, but a virtual mode from the harmonic excitation.
### Table 2: Estimation for the virtual mode from harmonic excitation while $f_h = 34.6$Hz

<table>
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<th>Model order</th>
<th>Estimated frequencies</th>
<th>Estimated damping ratios</th>
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<td>IV</td>
<td>SRA</td>
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### 5. CONCLUSIONS

This paper investigated the numerical performance of time-domain methods (SSI, SRA and IV) for stochastic modal identification in the presence of harmonic excitations. Theoretically, their numerical models and algorithms were summarized and the associated numerical conditioning and stability were compared. In the numerical study, the three methods were employed to extract the modal information for a simulated 5-DOF mass-damper-spring system which was excited by random Gaussian white noise and a harmonic component. Overall, the projection-driven SSI method was found to be superior to the covariance-driven IV and SRA methods. The structural information could be identified from the SSI method more easily and accurate than the IV and SRA methods. The mode associated with the harmonic component in the excitation was stabilized in the stability diagrams from all the three methods. However, only the estimated damping ratios from the SSI method are almost zero. A zero damping ratio could be utilized to confirm it is associated with a virtual mode from the harmonic excitation rather than a real structural mode.

From the numerical results of the scenario that the harmonic frequency was close to a modal frequency, the estimated modal parameters from the SRA method were highly perturbed and the stability diagrams associated with the SRA method were prone to produce stable spurious poles. Meanwhile, the IV method totally lost the ability to identify the structural mode whose frequency is close to the harmonic component. In contrast, the SSI method could accurately identify the harmonic frequency and the structural modal information. Thus, the IV and SRA methods without modification are not recommended.
if the harmonic frequency of the excitation is close to a structure’s modal frequency. The SSI method, even without “modification”, is a promising tool for the operational modal analysis in the presence of harmonic excitations.

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REFERENCES


