Predicting in-operation dynamics by assembly of measured components

Substructuring approaches have been used already in the sixties to reduce the cost of dynamic analysis of large models. In such numerical substructuring, models of components are reduced in size by defining a small representation space and assembling the components as super-elements. This paradigm of "divide and conquer" is also very attractive in experimental vibration, where the transfer functions of components are measured individually and where the dynamics of the system are obtained by assembling the measured components in a simulation model. Such experimental substructuring approaches have already been investigated in the seventies, but with limited success. Over the last ten years, the subject has received renewed attention thanks to recently proposed testing and data processing strategies, and thanks to the significant advances in sensors and data acquisition capabilities.

Component dynamics and operational excitations form the basis of the field of Transfer Path Analysis (TPA) which also includes the so-called Transmissibility-based TPA approaches, often used for troubleshooting. We will give an overview of different approaches to experimentally investigate the vibration of a system based on the measurements of its component, discussing techniques to build meaningful experimental models from measurements and explaining different ways to characterize the operational excitation sources. The experimental substructuring methodology will be illustrated using an industrial application.

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PREDICTING IN-OPERATION DYNAMICS BY ASSEMBLY OF MEASURED COMPONENTS

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\section*{ABSTRACT}
Transfer path analysis is a term regrouping a large number different of methods used in vibration tests to troubleshoot NVH problems or to build models based on the measured dynamics of components. Here we give a short summary of the different concepts and illustrate the operational TPA approach with a small academical test.

\textit{Keywords: experimental substructuring, blocked force, TPA.}

\section*{1. INTRODUCTION}
Substructuring approaches have been used already in the sixties to reduce the cost of dynamic analysis of large models. In such numerical substructuring, models of components are reduced in size by defining a small representation space and assembling the components as super-elements. This paradigm of “divide and conquer” is also very attractive in experimental vibration, where the transfer functions of components are measured individually and where the dynamics of the system are obtained by assembling the measured components in a simulation model. Such experimental substructuring approaches have already been investigated in the seventies, but with limited success (e.g. \textsuperscript{11,2}). Over the last ten years, the subject has received renewed attention thanks to recently proposed testing and data processing strategies, and thanks to the significant advances in sensors and data acquisition capabilities.

In this short paper, we will summarize some of the important ideas in Frequency-Based-Substructuring and discuss how it can be used in Transfer Path Analysis (TPA). In particular we will explain how the admittance of an assembly can be obtained from its components and how the source excitation can be indirectly characterized. At the end we will also discuss other TPA approaches that can be used for fast troubleshooting.

\section*{2. FREQUENCY BASED SUBSTRUCTURING}
Let us consider two components (substructures) $A$ and $B$ for which the admittances have been identified experimentally or obtained from a finite element model for instance (Figure 1). Calling $u_1^A$, the degrees
of freedom of interest (displacements, velocities or accelerations) inside component \( A \), \( \mathbf{u}_3^B \), those inside component \( B \), and \( \mathbf{u}_2^A = \mathbf{u}_2^B = \mathbf{u}_2 \) all degrees of freedom on the interface, the dynamic equation of the components considered as free bodies can be written as

\[
\begin{bmatrix}
\mathbf{u}_1^A \\
\mathbf{u}_2^A \\
\mathbf{u}_2^B \\
\mathbf{u}_3^B
\end{bmatrix} =
\begin{bmatrix}
\mathbf{Y}_{11}^A & \mathbf{Y}_{12}^A & 0 & 0 \\
\mathbf{Y}_{21}^A & \mathbf{Y}_{22}^A & 0 & 0 \\
0 & 0 & \mathbf{Y}_{22}^B & \mathbf{Y}_{23}^B \\
0 & 0 & \mathbf{Y}_{32}^B & \mathbf{Y}_{33}^B
\end{bmatrix}
\begin{bmatrix}
\mathbf{f}_1^A \\
\mathbf{f}_2^A \\
\mathbf{f}_2^B \\
\mathbf{f}_3^B
\end{bmatrix}
+ \begin{bmatrix}
0 \\
\lambda \\
-\lambda \\
0
\end{bmatrix}
\tag{1}
\]

where we denoted by \( \mathbf{Y} \) the admittance matrices in the frequency domain, \( \mathbf{f} \) the external forces and \( \lambda \) the internal forces on the interface which are determined by the interface compatibility condition

\[
\mathbf{u}_2^A = \mathbf{u}_2^B \tag{2}
\]

Substituting (1) into (2), it can be easily verified that the interface internal forces are determined by the so-called dual interface problem

\[
(\mathbf{Y}_{22}^A + \mathbf{Y}_{22}^B)\lambda = -\mathbf{Y}_{21}^A \mathbf{f}_1^A + \mathbf{Y}_{23}^B \mathbf{f}_3^B \tag{3}
\]

if we assume, for simplicity, that no external forces are applied on the interface. Solving (3) for \( \lambda \) and substituting in (1) one obtains the Frequency Response Function (FRF) of the assembled system. For instance, it can be seen that if a force is applied in \( A \), the output in \( B \) is given by

\[
\mathbf{u}_3^B = \mathbf{Y}_{32}^B (\mathbf{Y}_{22}^A + \mathbf{Y}_{22}^B)^{-1} \mathbf{Y}_{21}^A \mathbf{f}_1^A = \mathbf{Y}_{AB} 31 \mathbf{f}_1^A \tag{4}
\]

This last result shows how the transfer function of the assembled system can be obtained from the admittances of its components. Note that the admittance information for all interface degrees of freedom are needed for the assembly.

This methodology can be generalized to any number of components as long as their dynamics can be assumed to be linear (see for instance [3]): this formalism is known as the (Lagrange multiplier) Frequency-Based Substructuring method or FBS. The method can also be written in the time domain using impulse responses instead of FRFs [4].

The FBS (shortly summarized above for the simple case of the transfer function between an input in \( A \) and an output in \( B \)) is easy to program, but tricky to apply in practice, mainly for two reasons:

1. When computing the dynamic response using the dynamics of its components, no account is taken of the interface dynamics (contact stiffness and damping effects in the connection) since it was not included in the separate measurements. This issue can be tackled by measuring one of the components with an additional appendage that is attached to it in the same way as the real components will be coupled in the final setup. Such a device is called a transmission simulator [5, 6] or substitute. This substitute is usually a simple substructure for which the admittance is well-known (for
instance from a Finite Element model) and the effect of which will be eliminated in the assembly by subtracting its admittance. This can be easily achieved by applying the FBS formulas first with the negative admittance of the substitute, then applying it again to couple the admittance of the real receiver component. In this way the interface dynamics is properly accounted for (see for instance [7, 8]).

2. The second important issue arises due to the fact that the dually assembled admittance \((Y_{22}^A + Y_{22}^B)\) needs to be inverted during the coupling process (see for instance equation (4)). Since the admittances are obtained (at least for some parts) from measurements, noise and location or orientation inaccuracies make the components admittances non fully consistent. Since the admittances are usually badly conditioned, the FBS often results in wrong and meaningless estimations of the assembly admittance, typically exhibiting so-called spurious peaks at the resonance frequencies of the components [9]. This problem can be handled using (and often combining) two remedies:

- using a substitute (transmission simulator), as explained here above, not only allows accounting for the interface dynamics, but also allows exercising the interface in a manner much closer to its behavior in the assembly than if its interface would be left free during its characterization. The loading of the interface by the substitute will therefore induce a much better conditioning of the interface admittance.

- assuming a priori certain behavior modes for the interface region and measuring the FRFs for many locations close and on the interface, a weakened compatibility can be imposed. This allows alleviating the effect of local deformation that are badly measured. The easiest application of this ideas is the virtual point transformation [10, 11]: considering the components to be connected through locally rigid connections assumed to have only 3 translations and 3 rotations, a measured interface admittance of a component \(s\) can be reduced to the virtual point as

\[
Y_{22,vp}^s = T_u Y_{22}^s T_f^T
\]

where \(Y_{22,vp}^s\) is of dimension 6 x 6 whereas \(Y_{22}^s\) is the full admittance obtained from a highly overdetermined number of measurements in the interface region. The restriction matrix \(T_u\) (respectively \(T_f\)) transforms the interface responses (resp. forces) into average displacements and rotations (resp. resulting forces and moments), see for instance [7] for further details. By transforming the interface into a small number of generalized degrees of freedom, the FBS enforces compatibility only for those generalized motion, leaving uncoupled the motion that can not be represented by the assumed interface modes. This is an essential step for a successful coupling and can only be achieved thanks to the fact that the assembly is considered in a dual manner as in equation (1). Indeed, if a primal assembly of the impedance was considered, the interface compatibility could later not be weakened, resulting in a highly inaccurate representation of the interface dynamics.

As can be seen from the discussion above, building the model of an assembly using measured information of its component is based on simple mathematics but requires many precaution in practice. Examples of FBS applications to realistic and industrial structures can be found for instance in [12, 13, 7].

3. SOURCE CHARACTERISATION

When dealing with Noise, Vibration and Harshness problems, it is not sufficient to have a model of the assembly (resulting from a numerical model or from the FBS method explained above) but one also needs to characterize the excitation source. Here we shortly describe how the effect of the source can be properly characterized in practice.

Imagine that an excitation force is acting in a component \(A\) on the degrees of freedom denoted by 1 (refer to Figure 1) and let us assume that one is interested in engineering the vibration (or sound pressure) for
the degrees of freedom 3 in component B. Component A would be called a source component whereas substructure B would be considered as the receiver. In practice it is rarely possible to directly measure the excitation $f_1^A$ and often it cannot even be located precisely (e.g. gear contacts, electric motors ...). Therefore the excitation source needs to be characterized in an indirect manner, but in such a way that the estimated output $u_3^B$ is correct.

Therefore let us consider a fictitious pair of equal and opposite external forces of amplitude $F_2$ applied on the interface (see Figure 2). This obviously does not change the dynamics of the system. Let us then suppose that the forces $-F_2$ are exactly the forces needed to block all motion of the interface when the excitation source $f_1^A$ is operating, hence producing $u_2 = 0$. For the output $u_3^B$ the effect of the combination of $f_1^A$ and $-F_2$ is not perceived and they can be eliminated from the problem. We thus understand that the force $F_2$ is an equivalent fictitious force that has the same effect as the true excitation for what the receiver is concerned (obviously not for the source).

![Figure 2: Equivalent and blocked forces](image)

From the reasoning above, it is understood that this equivalent force is equal to minus the force needed to fix the interface, known as the blocked force: writing the dynamic equation for the true system and for the equivalent force case (left and right side of Figure 2),

\[
\begin{align*}
\mathbf{u}_3^B &= \mathbf{Y}_{31}^A f_1^A = \mathbf{Y}_{32}^B (\mathbf{Y}_{22}^A + \mathbf{Y}_{22}^B)^{-1} \mathbf{Y}_{21}^A f_1^A \\
\mathbf{u}_3^B &= \mathbf{Y}_{32}^B F_2 = \mathbf{Y}_{32}^B (\mathbf{Y}_{22}^A + \mathbf{Y}_{22}^B)^{-1} \mathbf{Y}_{22}^A F_2
\end{align*}
\]

and therefore the equivalent force is expressed by

\[
F_2 = \mathbf{Y}_{22}^A \mathbf{Y}_{21}^A f_1^A
\]

Further observing that

\[
\mathbf{u}_{2, \text{free}}^A = \mathbf{Y}_{21}^A f_1^A
\]

is the interface motion that would be measured if the source interface is left free (sometimes called free interface velocities by reference to acoustical problems [14, 11]), the equivalent force can also be written as

\[
F_2 = \mathbf{Y}_{22}^A \mathbf{u}_{2, \text{free}}^A
\]

The blocked force can be measured by clamping the source A on a fully rigid test rig and measuring the reaction forces on the rig. This however is difficult in practice and it might be easier to let the interface free, measure the free interface motion $\mathbf{u}_{2, \text{free}}^A$ and the admittance $\mathbf{Y}_{22}^A$, then apply formula [10, 11]. Leaving the interface completely free is also not always possible: if the blocked force is to be evaluated on a flexible test rig, one can measure both the force $g_2^R$ applied by the source on the test rig and the interface motion $\mathbf{u}_2^R$, and compute the blocked force as [11]

\[
F_2 = g_2^R + \mathbf{Y}_{22}^A \mathbf{u}_{2, \text{free}}^A
\]

A last and very useful manner to measure the equivalent (blocked) force is to directly consider its definition (7). Let us consider indicator locations close to the interface and denoted by 4 in B. Applying then (7) for those indicator sensors, one finds

\[
\mathbf{u}_4^B = \mathbf{Y}_{42}^A F_2
\]
Therefore if one measures enough indicator sensors as well as the admittance $Y_{AB}$ of the assembled system, one can compute the equivalent force as

$$F_2 = Y_{AB}^+ u_4$$

(13)

where $Y_{AB}^+$ is a pseudo inverse of $Y_{AB}$ resulting from a least square solution of (12), assuming that the number of indicators dofs is higher than the number of interface dofs. This easy technique, proposed in [15], is often well suited in practice since it does not require disassembling the structure and is known as the in situ approach.

To summarize, the vibration of a system can be analysed in the following steps:

1. measure (or obtain from numerical models) the admittances of $A$ and $B$, and build the admittance of the assembly according to the FBS formula, for instance (4).
2. measure the blocked force (equal to minus the equivalent force) according to one of the techniques discussed above.
3. predict the output from equation (7).

This methodology, known as Substructure-based TPA, allows predicting the output when the design of the receiver is modified. It is therefore very useful in design, but rather cumbersome if TPA is applied only for troubleshooting a noise and vibration problem. In that case, more straightforward techniques can be used as explained in the next section.

4. CLASSICAL TPA AND OPERATIONAL TPA

A common way to analyze the origin of noise and vibration perceived in a component is to measure the force reaching the receiver through its different interfaces. Quantifying then the relative contribution of the interfaces gives useful information to the engineer for fast troubleshooting. The interface force (not to be confounded with the equivalent force in the component-based TPA discussed in the previous section) can for instance be measured by placing force sensors between the components. The dynamics of the receiver is then simply described by considering equation (14) for the receiver part and assuming that no external force is applied on it:

$$u_3^B = Y_{32}^B (-\lambda)$$

(14)

where $-\lambda$ is the internal force measured by the force sensors (Figure 3, left). Such an approach, called classical TPA, requires only knowing the FRF $Y_{32}^B$ of the receiver and the interface force. It has however two main drawbacks:

- measuring the internal interface forces in operation is difficult since it requires installing force sensors between components, which is cumbersome and modifies the interface dynamics. In practice indirect measurement of the interface forces are used (using the stiffness of mounts or using a matrix inverse characterization with interface indicator sensors, see for instance [11]).
- following this approach, no accurate prediction can be done when the dynamics of the receiver is modified. Indeed, the measured internal interface force is a function of the component dynamics and will also change when the components are modified (see for instance equation (3)). Only the component-based TPA can properly predict the effect of modifications. Nevertheless the classical TPA approach is often used for fast troubleshooting.
In order to avoid the problem of measuring interface forces needed in the classical TPA, one can also consider the transmissibility between vibrations close to the interfaces and vibrations at the output of the receiver. This can give useful information to identify the origin of vibration problems. To explain this so-called transmissibility-based TPA let us define a transmissibility operator $T$ such that

$$u_3^B = T_{34}^B u_4^B$$

(15)

where $u_4^B$ are indicator sensors close to the interface (Figure 3, right). The transmissibility is a property of the receiver, independent of the excitation in the source, if there are enough indicator sensors to properly observe all degrees of freedom on the interface. If one measures the indicator vibrations $u_4^B$ and the outputs $u_3^B$ for different operational conditions, namely for different forces coming from the source, or for different time intervals during a random excitation, the relation (15) can be written for several realizations. In the frequency domain we can thus write

$$
\begin{bmatrix}
  u_3^{B(1)} \\
  u_3^{B(2)} \\
  \vdots
\end{bmatrix} =
T_{34}^B
\begin{bmatrix}
  u_4^{B(1)} \\
  u_4^{B(2)} \\
  \vdots
\end{bmatrix}
$$

(16)

which in matrix form can be written as

$$U_3^B = T_{34}^B U_4^B$$

(17)

Solving this equation to determine the transmissibility, we find

$$T_{34}^B = U_3^B [U_4^B]^+$$

(18)

where $[U_4^B]^+$ denotes a pseudo-inverse related to a least square solution of (17). A practical way to compute the pseudo-inverse is to apply a singular valued decomposition (SVD) such that

$$U_4^B = X_4 \Sigma_4 V_4^*$$

(19)

where $X_4$ and $V_4$ are orthonormal matrices ($^*$ indicating the Hermitian) and where $\Sigma_4$ is a diagonal matrix containing the singular values. The smallest singular values are amplifying the measurement noise in the channels $u_4$, and are thus usually not consired in the pseudo-inverse. Neglecting then the small singular values and considering only the parts $\tilde{X}_4$ and $\tilde{V}_4$ associated to the highest singular values, the transmissibility is obtained as

$$T_{34}^B \simeq U_3^B \begin{bmatrix} \tilde{V}_4 \Sigma_4^{-1} \tilde{X}_4^* \end{bmatrix}$$

(20)

Once the transmissibility has been estimated, one can use equation (15) to evaluate the contribution of different interface regions to the output. Further details on transmissibility-based TPA can for instance be found in [11, 16].

To illustrate this method (sometimes called operational TPA or OTPA), let us consider an aluminum L-shaped plate (longer sides having a length of 0.5 m, thickness of 1mm) excited on two corners by a set of three shakers producing uncorrelated random forces (Figure 4). One shaker excites a corner, interface 1, over a bracket (introducing a moment and a vertical force that are correlated). The opposite
corner, interface 2, is excited with two shakers over a bracket (introducing an uncorrelated moment and vertical force). The plate is hanging in a support made of soft rubber strings. The interface indicators are provided by 4 one-dimensional accelerometers close to each interface, measuring the out-of-plane vibration of the plate (Figure 4). The output is located between the two interface regions in the middle of the plate.

The signals are measured for 40 seconds with a rate of 1638 samples per second and divided in 16 time slots of 2.5 seconds. The time signals are then transformed in the frequency domain (Fourier Transform with Hanning window). This results in $U_{3B}$ and $U_{4B}$ of dimension $1 \times 16$ and $8 \times 16$ respectively for each frequency with a frequency resolution of 0.4 Hz.

In a first experiment, only shaker 1 is activated. The singular values $\Sigma_{3B}$ of $U_{3B}$ over the frequency range are plotted in Figure 5. Clearly one dominant singular value can be recognized over the frequency range due to the fact that all the indicator signals are originating from a single source. Applying then equation (20) to compute the transmissibility, one can reconstruct the output using equation (17). Comparing then the reconstructed output $U_{3rec} = T_{34B}U_{4B}$ obtained with the estimated transmissibility and the original measurement $U_{3B}$, one observes that using only one singular value leads to a nearly perfect match (Figure 5 right), showing that indeed only one general interface contribution can be considered for this case.
To evaluate the contribution of both interfaces, one can construct the partial output generated by all indicators belonging to a given interface:

\[
u_3^B = u_{3\text{int}1}^B + u_{3\text{int}2}^B = T_{34\text{int}1}^B u_{4\text{int}1}^B + T_{34\text{int}2}^B u_{4\text{int}2}^B
\]  

(21)

where the subscript $4_{\text{int}j}$ indicates the subset of indicators belonging to an interface $j$. In figure 6, one sees that interface 1 (on which the shaker was active) clearly contributes the most. However one observes that the contribution of interface 2 is not very small. This is due to the fact that the indicators on interface 2 also exhibit vibrations due to the fact that they are not far away from interface 1 and will therefore also be excited. Note that the same problem would arise if a classical TPA would have been used for this experiment: the internal interface forces in interface 2 are not null and therefore, although the contribution of this interface would be estimated to be small, it would not be very small compared to the contribution of interface 1. If however equivalent forces would be estimated for this case (for instance using the in-situ approach described by equation (13)), the dynamic coupling between interfaces would be accounted for and an equivalent force would be found only on interface 1 (at least in theory).

If now all 3 shakers are active, the singular values of $U_4^B$ show that 3 dominant contributions must be considered (Figure 7) to build the transmissibility matrix. Computing then the contribution of both interfaces, it is seen that both interfaces are now estimated to contribute equally strongly to the output (Figure 8).
5. CONCLUSION

We have presented a short overview of different transfer path analysis (TPA) approaches, arguing that the component-based TPA is the only approach allowing for proper estimation of the dynamics of the system after modification, which is necessary in a design and optimization process. This methodology is cumbersome because it is based on the admittances of the components and on an equivalent interface force (blocked force). It also requires special attention when used in practice.

For rapid troubleshooting, techniques like classical TPA (where internal interface forces are considered) and transmissibility-based TPA (like the OTPA) can be used to rapidly estimate the importance of interface contributions. However, the interpretation of the information resulting from these techniques require good engineering judgement as illustrated with a small experiment in this paper.

REFERENCES


