THE EFFECTS OF STEP, RAMP AND SINUSOIDAL FORCES ON RESPONSE OF MULTI STEP TIMOSHENKO BEAM

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Abstract

Flexible beams are generally modeled according to the classical Euler-Bernoulli theory. This formulation provides a good description of the dynamical behavior of the system if the beam's cross sectional dimension is small in comparison of its length. In this case, the effects of the rotary inertia of the beam are not considered. A more accurate beam model is provided by the Timoshenko theory, according to which the rotary inertia and also the deformation due to shear are considered. The resulting Timoshenko model of the beam is generally more accurate in predicting the beam's response than the Euler-Bernoulli one. In this work, an exact approach for forced vibration analysis of a multi-step beam with an arbitrary number of steps (cross sectional change) and boundary conditions has been used. Using the fundamental solutions and recurrence formulas developed in literature, the forced responses of a multi step beam (Timoshenko and Euler Bernoulli) with an arbitrary number of steps can be easily determined. The main advantage of the work is that the forced responses of the beam with any kind of two end supports, any finite number of cross sectional change and any kind of external force can be conveniently determined. Sample responses are given to illustrate the differences between Timoshenko and Euler Bernoulli models and to study the effect of cross sectional changes on the forced responses of multi step beams.

1 Introduction

Timoshenko and Euler Bernoulli (EB) beams are extensively used in literature for modeling beams. An exact formulation of the beam problem was first investigated in terms of general elasticity equations by Pochhammer (1876) and Chree (1889). (Ref [1]) They derived the equations that describe a vibrating solid cylinder. However they were not practical because they usually result in lots of complicated equations which mostly are indissoluble. (Ref [2]) It was recognized by the early researchers that the bending effect is the single most important factor in a transversely vibrating beam. The Euler Bernoulli model includes the strain energy due to the bending and the kinetic energy due to the lateral displacement. The Euler Bernoulli model dates back to the 18th century. Jacob Bernoulli (1654-1705) first discovered that the curvature of an elastic beam at any point is proportional to the bending moment at that point. Daniel Bernoulli (1700-1782), nephew of Jacob, was the first one who formulated the differential equation of motion of a vibrating beam.

Later, Jacob Bernoulli's theory was accepted by Leonhard Euler (1707-1783) in his investigation of the shape of elastic beams under various loading conditions. Many advances on the elastic curves were made by Euler. (Ref [3]) The Euler Bernoulli beam theory, sometimes called the classical beam theory, Euler beam theory, Bernoulli beam theory, or Bernoulli Euler beam theory, is the most commonly used because it is simple and provides reasonable engineering approximations for many problems. However, the Euler Bernoulli model tends to slightly overestimate the natural frequencies. This problem is exacerbated for the natural frequencies of the higher modes. Also, the prediction is better for slender beams than non-slender beams. Mostly recent works focus on the effect of crack on beams behavior and rarely concentrate on step change by itself such as C. Mei et al who worked on cracked beam with on step change in cross sectional area. (Ref [4]) In 2001 S.P. Lee et al. MAITI modeled transverse vibrations of short beams for crack detection ref [5] and in 2004 natural frequencies of bending vibration of Timoshenko cracked beams were obtained by J. A. Loya and his colleague. (Ref [6]) In 2002, S. Naguleswaran worked on Vibration and stability of an Euler–Bernoulli beam with up to three-step changes in cross-section and in axial force ref [7] and in 2005 Xing-Jian Dong et al developed Vibration analysis of a stepped laminated composite Timoshenko beam. (Ref [8])

2 Mathematic Formulation for Timoshenko Beam

Detailed derivations for the Euler-Bernoulli model can be found in text books ref [9-14]. Timoshenko proposed a beam theory which adds the effects of shear distortion and rotary inertia to the Euler-Bernoulli models. Derivation of Timoshenko equations of motion is not main concern in this work and they will be presented just for the sake of tight knit between subjects. (Ref [2]) The PDE describing motional behavior of Timoshenko beam are in this way:

$$\rho A \frac{\partial^2 v(x,t)}{\partial t^2} - KGA\left(\frac{\partial^2 v(x,t)}{\partial x^2} - \frac{\partial \alpha(x,t)}{\partial x}\right)^2 = f(x,t) \tag{1}$$
$$\rho I \frac{\partial^2 \alpha(x,t)}{\partial t^2} - \frac{\partial^2 \alpha(x,t)}{\partial x^2} - KGA\left(\frac{\partial v(x,t)}{\partial x} - \alpha(x,t)\right) = 0$$

Where ρ, A, I, G, K denote density, cross sectional area, moment of inertia, shear modulus and shape factor respectively. Furthermore, v is deflection and α denotes slop and $f(x,t)$ is the applied force which is function of time and place. In order to solve the homogeneous problem, the forcing function is set to zero. The answers for homogeneous equation of Eq. (1) would be:

$$\begin{pmatrix} W(x) \\ \Psi(x) \end{pmatrix} = \begin{pmatrix} C_1 \\ D_1 \end{pmatrix} Sin(r_1 x) + \begin{pmatrix} C_2 \\ D_2 \end{pmatrix} Cos(r_1 x) + \begin{pmatrix} C_3 \\ D_3 \end{pmatrix} Sinh(r_2 x) + \begin{pmatrix} C_4 \\ D_4 \end{pmatrix} Cosh(r_2 x) \tag{2}$$

Which has got hyperbolic form and the parameters used in it are described in this way:

$$r_1 = \sqrt{\left(I + \frac{1}{KG}\right)\frac{\rho\omega^2}{2} + \sqrt{\left(I + \frac{1}{KG}\right)^2 \left(\frac{\rho\omega^2}{2}\right)^2 + \rho A \omega^2}} \tag{3}$$
$$r_2 = \sqrt{-\left(I + \frac{1}{KG}\right)\frac{\rho\omega^2}{2} + \sqrt{\left(I + \frac{1}{KG}\right)^2 \left(\frac{\rho\omega^2}{2}\right)^2 + \rho A \omega^2}}$$

The relation between the coefficients of $W(x)$ and $\Psi(x)$ is available in literature. Furthermore, boundary conditions at steps are:

$$\begin{aligned} W_L &= W_R \\ \Psi_L &= \Psi_R \\ M_L &= M_R \\ V_L &= V_R \end{aligned} \tag{4}$$

Where, W, Ψ, M and V denote deflection, slope, momentum, and shear force in left and right sides of steps.

3 The Forced Responses of Timoshenko Model

In order to obtain the forced response of the beam, eigenfunction expansion method has been used. Therefore, the orthogonallity conditions of the eigenfunctions have to be established for Timoshenko beam model, and the related equations could be found in Ref. [2]. Here, the results have been used in order to obtain forced responses of the beam. In some works other methods such as boundary integral formulation has been used [15]. For the model the spatial equations of the homogeneous problem can be written using the operator formalism

$$L(W_n) = \omega_n^{\,2} M(W_n) \tag{5}$$

Where W_n can denote the n_{th} vector of eigenfunctions $[W_n, \Psi_n]^T$ for the Timoshenko model, and corresponds to the natural frequency $\omega_n^{\,2}$ uniquely to within an arbitrary constant. The expressions for the operators for Timoshenko model is given below

$$L(W_n) = \begin{pmatrix} KGA\dfrac{d^2}{dx^2} & -KGA\dfrac{d}{dx} \\ KGA\dfrac{d}{dx} & \dfrac{d^2}{dx^2} - KGA \end{pmatrix} \begin{pmatrix} W_n \\ \Psi_n \end{pmatrix} \tag{6}$$

$$M(W_n) = \begin{pmatrix} \rho A & 0 \\ 0 & \rho I \end{pmatrix} \begin{pmatrix} W_n \\ \Psi_n \end{pmatrix}$$

The eigenfunctions will be normalized by setting the integral equal to one,

$$\int_0^1 W_n^{\,T} M(W_m) dx = 1 \tag{7}$$

$$n = 1,2,3,4,\ldots$$

For the systems we consider, the mentioned operators are self-adjoint and the eigenfunctions are orthogonal to each other. The method of eigenfunction expansion assumes that the solutions $v(x,t)$ to equations of motion given in Eq. (1) and the forcing function $f(x,t)$ can be represented as a summation of eigenfunctions multiplied by functions of time that are to be determined, that is:

$$v(x,t) = \sum_{n=1}^{\infty} \eta_n(t) W_n(x) \tag{8, 9}$$

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) M(W_n(x))$$

Where assuming:

$$\eta_n(t)=\int_0^1 W_m{}^T M(v(x,t))dx \tag{10}$$

$$F_m(t)=\int_0^1 W_m{}^T f(x,t)dx \tag{11}$$

And substituting the assumed solution, Eq. (8), and the forcing function, Eq. (9), into Eq. (1), then

$$\sum_{n=1}^{\infty}\frac{d^2\eta_n(t)}{dt^2}M(W_n(x))+\sum_{n=1}^{\infty}\eta_n(t)L(W_n(x))=\sum_{n=1}^{\infty}F_n(t)M(W_n(x)) \tag{12}$$

where the expressions for the operators M and L¸ are given in Eq. (6) respectively. Using Eq. (5), the last equation becomes

$$\sum_{n=1}^{\infty}\left[\frac{d^2\eta_n(t)}{dt^2}+\omega_n{}^2\eta_n(t)\right]M(W_n(x))=\sum_{n=1}^{\infty}F_n(t)M(W_n(x)) \tag{13}$$

Multiplying by Wm and integrating over the domain $0 \le x \le 1$ results in

$$\frac{d^2\eta_m(t)}{dt^2}+\omega_m{}^2\eta_m(t)=F_m(t) \tag{14}$$

where the solution is given by

$$\eta_m(t)=\frac{1}{\omega_m}\int_0^t F_m(t)Sin(\omega_m(t-\tau))d\tau+\eta_m(0)Cos(\omega_m t)+\frac{1}{\omega_m}\left.\frac{d\eta_m}{dt}\right|_{t=0} Sin(\omega_m t) \tag{15}$$

F_m is given by Eq. (11), and $\eta_m(0)$ and $\frac{d\eta_m(0)}{dt}$ are obtained from the initial conditions, $v(x,0)$ and $\dot{v}(x,0)$, using Eq. (10),

$$\eta_m(0)=\int_0^1 W_m{}^T M(v(x,0))dx \qquad \dot{\eta}_m(t)=\int_0^1 W_m{}^T M(\dot{v}(x,t))dx \tag{16}$$

Finally, the solution is given by

$$v(x,t)=\sum_{n=1}^{\infty}\eta_n(t)W_n(x) \tag{17}$$

and η_n is given by Eq. (15).

4 Sample Responses of Timoshenko Beam

In this section force response of the Timoshenko and Euler Bernoulli models when subjected to step, ramp and sinusoidal loads (Figure 2) will be analyzed and deflection at different points of the beam will be computed. The example beams are 3 and 4 step models (Figure 1) with physical characteristics of E = 200GPa, G = 75.8GPa, ν = .29, ρ = 7860Kg/m³, Width (w) = 0.0127m, and Length (L) = 0.1905m. Furthermore, the cross sectional changes are of 5% reduction in thickness from left to right for each step; for instance, 0.95h, 0.9h and so on.

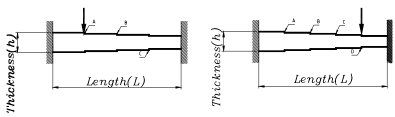

Figure.1 Three and four step clamped-clamped models

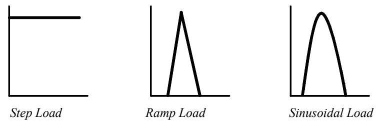

Figure.2 Step, ramp and sinusoidal loads

Figure.3. Deflection at node A of 4-step CC beam when subjected to step load of 1000N at node A when thickness of beam (h) changes from 3.97mm to 5.56mm. (Left picture for when thickness is 3.97mm and right picture for when thickness is 5.56mm) It seems that the step load affects fourth mode eigenfrequency in Timoshenko Model———— . Euler Bernoulli Model———— .

Figure.4. Deflection at node B of 4-step CC beam when subjected to step load of 1000N at node A when thickness of beam (h) changes from 3.97mm to 5.56mm. (Left picture for when thickness is 3.97mm and right picture for when thickness is 5.56mm) It seems that the step load affects fourth mode eigenfrequency in Timoshenko Model———— . Euler Bernoulli Model———— .

Figure.5. Deflection at node C of 4-step CC beam when subjected to step load of 1000N at node A when thickness of beam (h) changes from 3.97mm to 5.56mm. (Left picture for when thickness is 3.97mm and right picture for when thickness is 5.56mm) It seems that the step load affects fourth mode eigenfrequency in Timoshenko Model———— . Euler Bernoulli Model———— .

Figure.6. Deflection at node D of 4-step CC beam when subjected to step load of 1000N at node A when thickness of beam (h) changes from 3.97mm to 5.56mm. (Left picture for when thickness is 3.97mm and right picture for when thickness is 5.56mm) It seems that the step load affects fourth mode eigenfrequency in Timoshenko Model——— . Euler Bernoulli Model——— .

Figure.7. Deflection at nodes A and B of 3-step CC Timoshenko beam with thickness of 5.56mm when it is subjected to a step load of 100N at 0.05m from left side.

Figure.8. Deflection at nodes A and B of 3-step CC Timoshenko beam with thickness of 5.56mm when it is subjected to a ramp load of 100N at 0.05m from left side with duration of 0.003 sec.

Figure.9. Deflection at nodes A and B of 3-step CC Timoshenko beam with thickness of 5.56mm when it is subjected to a sinusoidal load of 100N at 0.05m from left side with duration of 0.2 sec.

5 Conclusion

Timoshenko and Euler Bernoulli models were used to determine responses of multi step beams with step, ramp and sinusoidal loads on them. Forced responses are illustrated for a multi step beam of clamped-clamped boundary condition. The responses are efficient for handling cases with arbitrary number of cross sectional changes as the characteristic matrix has been solved at Mathematica environment. As the combination of the forces is approached, responses of the beam can also be computed. Comparisons of the Timoshenko beam results with Euler Bernoulli analyses for clamped-clamped containing up to four steps have shown good agreement. The work is also extended to include the effect of different kinds of loads, which are important for stability and vibration analysis of beam. The numerical studies conducted on 4 and 3 step beams showed that deflection changes due to step loads are important. Regarding Timoshenko beam's response step loads of about 100 N are found to affect forth mode eigenfrequencies. This effect is observed to be less significant for the forth mode eigenfrequenceies among the Euler Bernoulli beam's responses. The presence of sinusoidal forces is found to significantly increase the deflection depending on the locations and severity of them. In addition, it is shown that load locations and severities can significantly affect the changes in responses. For the 3 step case, when the sinusoidal load is applied on point 0.05 from left side, nodes A and B can result in different displacements. The use of the proposed work is believed to provide efficient results that can be used in damage identification studies by considering step, ramp and sinusoidal loads. It can also serve as verification for numerical methods developed for damage detection purposes. Further researches required to experimentally examine the effects of step changes on the responses of multi steps beams. In this way, it is possible to show the validity of these responses for cross sectional changes adopted in the analyses when clamped-clamped boundary condition is present.

6 References

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